Computational Science:
Computational Methods in Engineering

Preconditioning

Condition Number

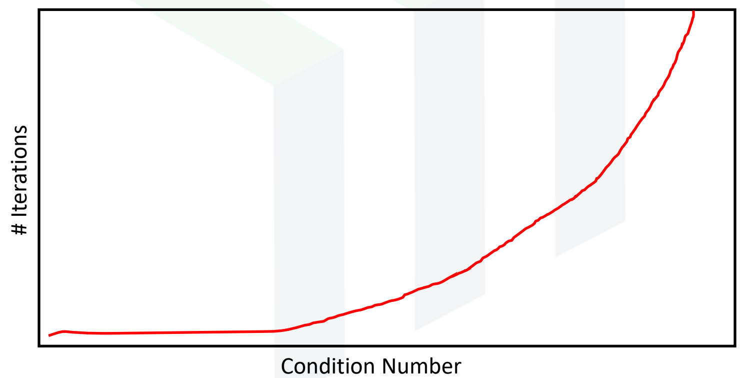
Condition number of a matrix $[A]$ measures how much the answer to a linear algebra problem changes due to small changes in $[A]$.

The condition number can be calculated from the norms of the matrix.
The condition number of a matrix $[A]$ is defined as $\text{cond}(A) = \|A\|\|A^{-1}\|$

A large condition number means the system is **ill conditioned** and a small condition number means the system is **well conditioned**.

Problem with ill-Conditioned Matrices

The number of iterations needed for an iterative solver to find a solution to $[A][x] = [b]$depends heavily on the condition number of the matrix $[A]$.



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Preconditioning

Suppose $[A][x] = [b]$ is to be solved iteratively, but $[A]$ has a high condition number.

A preconditioner $[P]$ is a nonsingular matrix such any of the following have a smaller condition number than $[A]$ alone.

$$[P]^{-1}[A] \quad \text{Left Preconditioned}$$

$$[A][P]^{-1} \quad \text{Right Preconditioned}$$

$$[P]^{-1}[A]\left([P]^{-1}\right)^{\mathrm{T}} \quad \text{Split Preconditioned}$$

The ideal preconditioner transforms $[A]$ into an identity matrix and makes any iterative method converge in one iteration.

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Right Preconditioning

Formulation

Original Equation

$$[A][x]=[b]$$

Insert Preconditioner

$$[A][P]^{-1}[P][x]=[b]$$

Derive Preconditioned Equation

$$[A'][y]=[b]$$

$$[A']=[A][P]^{-1} \qquad [y]=[P][x]$$

Implementation

1. Solve $[y]=[A']^{-1}[b]$
2. Calculate $[x]=[P]^{-1}[y]$

Iterative Solvers

1. Least-squares (LSQR)
2. Transpose-free quasi-minimal residual (TFQMR)
3. Stabilized biconjugate gradients (BICGSTAB)
4. Stabilized biconjugate gradients (l) (BICGSTABL)
5. Conjugate gradient square (CGS)

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Left Preconditioning

Formulation

Original Equation

$$[A][x]=[b]$$

Insert Preconditioner

$$[P]^{-1}[A][x]=[P]^{-1}[b]$$

Derive Preconditioned Equation

$$[A'][x]=[b']$$

$$[A']=[P]^{-1}[A] \qquad [b']=[P]^{-1}[b]$$

Implementation

1. Solve $[x]=[A']^{-1}[b']$

Iterative Solvers

1. Generalized minimum residual (GMRES)
2. Quasi-minimal residual (QMR)
3. Biconjugate gradients (BICG)

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Split Preconditioning

Formulation

Original Equation

$$[A][x]=[b]$$

Insert Preconditioner

$$[P]^{-1}[A][P]^{-T}[P]^{T}[x]=[P]^{-1}[b]$$

Derive Preconditioned Equation

$$[A'][y]=[b']$$

$$[A']=[P]^{-1}[A][P]^{-T}$$

$$[y]=[P]^{T}[x]$$

$$[b']=[P]^{-1}[b]$$

Implementation

1. Solve $[y]=[A']^{-1}[b']$
2. Calculate $[x]=[P]^{-T}[y]$

Iterative Solvers

1. Preconditioned conjugate gradients (PCG)
2. Minimum residual (MINRES)
3. Symmetric LQ (SYMMLQ)

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Jacobi (or Diagonal) Preconditioner

Perhaps the simplest preconditioner is the Jacobi preconditioner. In this case, the preconditioner $[P]$ is simply the diagonal of the matrix $[A]$.

$$[P]=\operatorname{diag}([A])$$

$$[A]=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \\ a_{N1} & a_{N2} & & a_{NN} \end{bmatrix}$$

$$[P]=\begin{bmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{NN} \end{bmatrix}$$

Jacobi preconditioning is particularly effective for diagonally dominant matrices.

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Notes About Preconditioning

- The preconditioner $[P]$ is chosen to accelerate the convergence of an iterative method.
- Preconditioning is needed when little progress is made between iterations or the residual error of an iterative solution stagnates.
- When the numbers in $[A]$ have massively different magnitudes (i.e. 2-3 orders of magnitude or more), $[A]$ will typically require a preconditioner.
- Incomplete factorizations, such as the incomplete LU factorization or the incomplete Cholesky factorization, are often used as preconditioners when natural preconditioners are absent.