Computational Science: Computational Methods in Engineering

Simpson’s Rules for Numerical Integration

Outline

• Simpson’s 1/3 Rule
• Simpson’s 3/8 Rule
Simpson’s 1/3 Rule

Suppose three discrete points of a function are known and these are fit them to a second-order polynomial.

\[ f(x) \approx a_0 + a_1 x + a_2 x^2 \]

Now integrate the polynomial under the curve.

\[
\int_{x_0}^{x_1} f(x) \, dx \approx \int_{x_0}^{x_1} (a_0 + a_1 x + a_2 x^2) \, dx \\
\approx \frac{1}{3} \Delta x (f_1 + 4f_2 + f_3)
\]

To implement Simpson’s 1/3 rule, simply apply this to \( f(x) \) in groups of 3 points.
Derivation of Simpson’s 1/3 Rule

First, fit the three points to a polynomial.

\[ f(x) \approx a_0 + a_1 x + a_2 x^2 \]

\[ a_0 = f_2 \quad a_1 = \frac{f_3 - f_1}{2\Delta x} \quad a_2 = \frac{f_3 - 2f_2 + f_1}{2(\Delta x)^2} \]

Second, integrate the polynomial from \(-\Delta x\) to \(\Delta x\).

\[ \int_{-\Delta x}^{\Delta x} \left( a_0 + a_1 x + a_2 x^2 \right) dx = \left[ a_0 x + \frac{1}{2} a_1 x^2 + \frac{1}{3} a_2 x^3 \right]_{-\Delta x}^{\Delta x} = 2a_0 \Delta x + \frac{2}{3} a_2 (\Delta x)^3 \]

Substitute in the expressions for \(a_0\), \(a_1\), and \(a_2\).

\[ 2a_0 \Delta x + \frac{2}{3} a_2 (\Delta x)^3 = 2f_2 \Delta x + \frac{2}{3} f_3 - 2f_2 + f_1 (\Delta x)^3 = \frac{1}{3} \Delta x (f_1 + 4f_2 + f_3) \]

Implementation of Simpson’s 1/3 Rule

Animation of Numerical Integration Using Simpson’s 1/3 Rule
Simpson’s 3/8 Rule

This is similar to Simpson’s 1/3 rule, except $f(x)$ is fit to a polynomial in groups of 4 points.

$$\int_{x_i}^{x_i + 4} f(x) \, dx \approx \frac{3}{8} \Delta x (f_1 + 3f_2 + 3f_3 + f_4)$$