Computational Science:
Computational Methods in Engineering

Solution Approach for Analyzing Transmission Lines

Maxwell's Equations

Maxwell's equations describes electromagnetic fields and what induces them.

$\nabla \bullet \vec{D} = \rho_v$	Gauss' law	$\vec{E} \equiv$ electric field intensity (V/m)
$\nabla \bullet \vec{B} = 0$	Gauss' law for magnetic fields	$\vec{D} \equiv$ electric flux density (C/m^2)
$\nabla \times \vec{E} = -\partial \vec{B} / \partial t$	Faraday's law	$\vec{H} \equiv$ magnetic field intensity (A/m)
$\nabla \times \vec{H} = \vec{J} + \partial \vec{D} / \partial t$	Ampere's circuit law	$\vec{B} \equiv$ magnetic flux density (Wb/m^2)
		$\rho_v \equiv$ volume charge density (C/m^3)
		$\vec{J} \equiv$ electric current density (A/m^2)

The constitutive relations describe how the electromagnetic fields interact with matter.

$\vec{D} = \varepsilon \vec{E}$	Electric response	$\varepsilon \equiv$ permittivity (F/m)
$\vec{B} = \mu \vec{H}$	Magnetic response	$\mu \equiv$ permeability (H/m)

Physical Constants

The permittivity and permeability are usually expressed in terms of the relative parameters.

$$\varepsilon = \varepsilon_0 \varepsilon_r$$ Permittivity

$$\mu = \mu_0 \mu_r$$ Permeability

$\varepsilon \equiv$ permittivity (F/m)
$\varepsilon_0 \equiv$ free space permittivity (F/m)
$\varepsilon_r \equiv$ relative permittivity (no units)

$\varepsilon_0 = 8.8541878176 \times 10^{-12}$ F/m
$1 \leq \varepsilon_r < \infty$

ε_r is commonly called the dielectric constant.

$\mu_0 = 1.2566370614 \times 10^{-6}$ H/m
$1 \leq \mu_r < \infty$

$\mu \equiv$ permeability (H/m)
$\mu_0 \equiv$ free space permeability (H/m)
$\mu_r \equiv$ relative permeability (no units)

The speed of light in free space can be calculated from the permittivity and permeability as follows.

$$c_0 = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 299,792,458 \text{ m/s}$$

$c_0 \equiv$ speed of light in vacuum (m/s)

EMPossible 3

Electrostatic Approximation

Transmission lines are waveguides. To be rigorous, they should be modeled as such. This can be rather computationally intensive. An alternative is to analyze transmission lines in the electrostatic approximation using Laplace's equation.

The dimensions of a transmission are typically much smaller than the operating wavelength so the wave nature is less important to consider. Therefore, Maxwell's equations can be solved assuming static fields.

$$\frac{\partial}{\partial t} \rightarrow 0 \quad \Rightarrow \quad \begin{aligned} &\nabla \bullet \vec{B} = 0 \\ &\nabla \bullet \vec{D} = \rho_v \\ &\nabla \times \vec{H} = \vec{J} + \cancel{\partial \vec{D} / \partial t} \\ &\nabla \times \vec{E} = \cancel{-\partial \vec{B} / \partial t} \end{aligned} \quad \Rightarrow \quad \begin{aligned} &\nabla \bullet \vec{B} = 0 \\ &\nabla \bullet \vec{D} = \rho_v \\ &\nabla \times \vec{H} = \vec{J} \\ &\nabla \times \vec{E} = 0 \end{aligned}$$

EMPossible 4

Electric Potential V

The vector electric field $\vec{E}$ and the scalar electric potential V describe the same physical phenomenon. They are just two mathematical forms of the same things. The electric potential allows equations to be solved in terms of a scalar quantity instead of a vector quantity.

In electrostatics, Faraday's law reduces to $\nabla \times \vec{E} = 0$.

From vector calculus, the curl of a gradient is always zero, $\nabla \times (\nabla V) = 0$.

Based on this, if a vector field does not have any curl, it must be possible to express it as the gradient of a scalar field.

$$\vec{E} = -\nabla V$$

The negative sign is incorporated to be consistent with the sign convention used with charges and fields.

$$\boxed{\vec{E} = -\nabla V}$$

$V \equiv \text{electric potential (volts)}$

Inhomogeneous Laplace's Equation

Away from charges ρ_v, the divergence condition for the electric field is

$$\nabla \bullet \vec{D} = 0 \qquad \text{Eq. (1)}$$

But $\vec{D} = \varepsilon \vec{E}$, so Eq. (1) can be written in terms of just $\vec{E}$.

$$\nabla \bullet \left(\varepsilon_r \vec{E} \right) = 0 \qquad \text{Eq. (2)}$$

Recall that $\varepsilon = \varepsilon_0 \varepsilon_r$, but ε_0 is a constant and drops from the equation.

The electric field $\vec{E}$ is related to the scalar potential V as follows.

$$\vec{E} = -\nabla V \qquad \text{Eq. (3)}$$

The inhomogeneous Laplace's equation is derived by substituting Eq. (3) into Eq. (2).

$$\boxed{\nabla \bullet \left[\varepsilon_r \left(\nabla V \right) \right] = 0}$$

Homogeneous Laplace's Equation

When the transmission line is embedded in a homogeneous dielectric, the relative permittivity ε_r is not a function of position and drops out of Laplace's equation.

$$\nabla \bullet \left[\varepsilon_r \left(\nabla V \right) \right] = 0$$
$$\varepsilon_r \nabla \bullet \left[\left(\nabla V \right) \right] = 0$$
$$\nabla \bullet \left[\left(\nabla V \right) \right] = 0$$
$$\nabla^2 V = 0$$

$$\boxed{\nabla^2 V = 0}$$

EMPossible

7

Distributed Capacitance C

In the electrostatic approximation, the transmission line is a capacitor. The total energy U stored in a capacitor is the total energy in the fields $\vec{D}$ and $\vec{E}$:

$$U = \tfrac{1}{2} \iint_A \left(\vec{D} \bullet \vec{E} \right) dA$$

The integral is taken over the entire cross section A of the device. For open devices like microstrips, this is an infinite area. In practice, integration happens over a large enough area to incorporate as much of of the electric field as possible.

From circuit theory, the capacitance C is related to the total stored energy U through

$$U = \frac{C V_0^2}{2}$$

V_0 is the voltage across the capacitor.

If the above equations are set equal, an equation is derived that calculates the distributed capacitance C from the electric field functions $\vec{D}$ and $\vec{E}$.

$$\boxed{C = \frac{1}{V_0^2} \iint_A \left(\vec{D} \bullet \vec{E} \right) dA}$$

EMPossible

8

Distributed Inductance L

The voltage along the transmission line embedded in a homogeneous material travels at the same velocity as an electromagnetic wave in that same medium.

$$v_V = v_E \quad \rightarrow \quad \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\mu_\mathrm{h}\varepsilon_\mathrm{h}}} \quad \rightarrow \quad LC = \frac{\mu_\mathrm{r,h}\varepsilon_\mathrm{r,h}}{c_0^2}$$

Solving this equation for L gives

$$L = \frac{\mu_\mathrm{r,h}\varepsilon_\mathrm{r,h}}{c_0^2 C}$$

This means that for transmission lines embedded in homogeneous materials, the distributed inductance L can be calculated directly from the distributed capacitance C.

Dielectric materials should not alter the inductance. However if the value of C calculated previously is used it will. This is incorrect. The solution is to calculate distributed capacitance with air dielectric C_h and then calculate the distributed inductance L from this.

$$\boxed{L = \frac{1}{c_0^2 C_\mathrm{h}}}$$

EMPossible 9

Calculating Transmission Line Parameters

The characteristic impedance Z_c is calculated from the distributed inductance L and distributed capacitance C through

$$\boxed{Z_\mathrm{c} = \sqrt{\frac{L}{C}}}$$

The velocity v of a signal travelling along the transmission line is

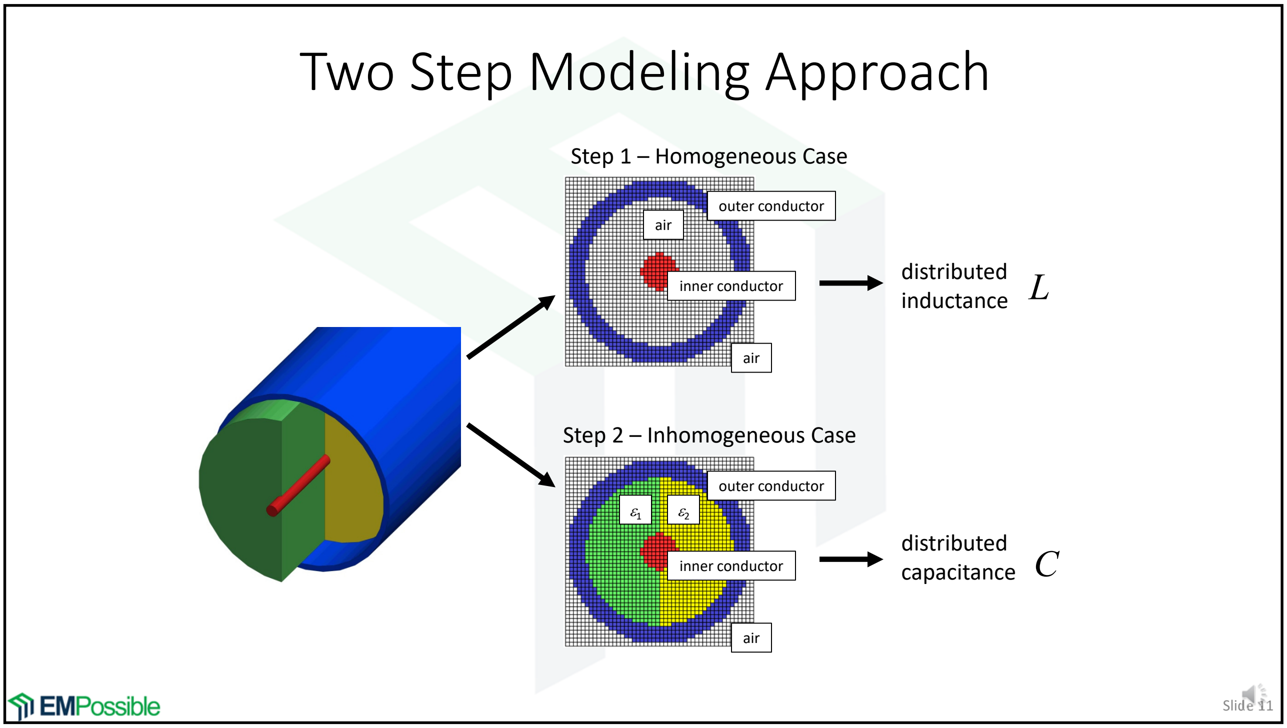
$$v = \frac{1}{\sqrt{LC}}$$

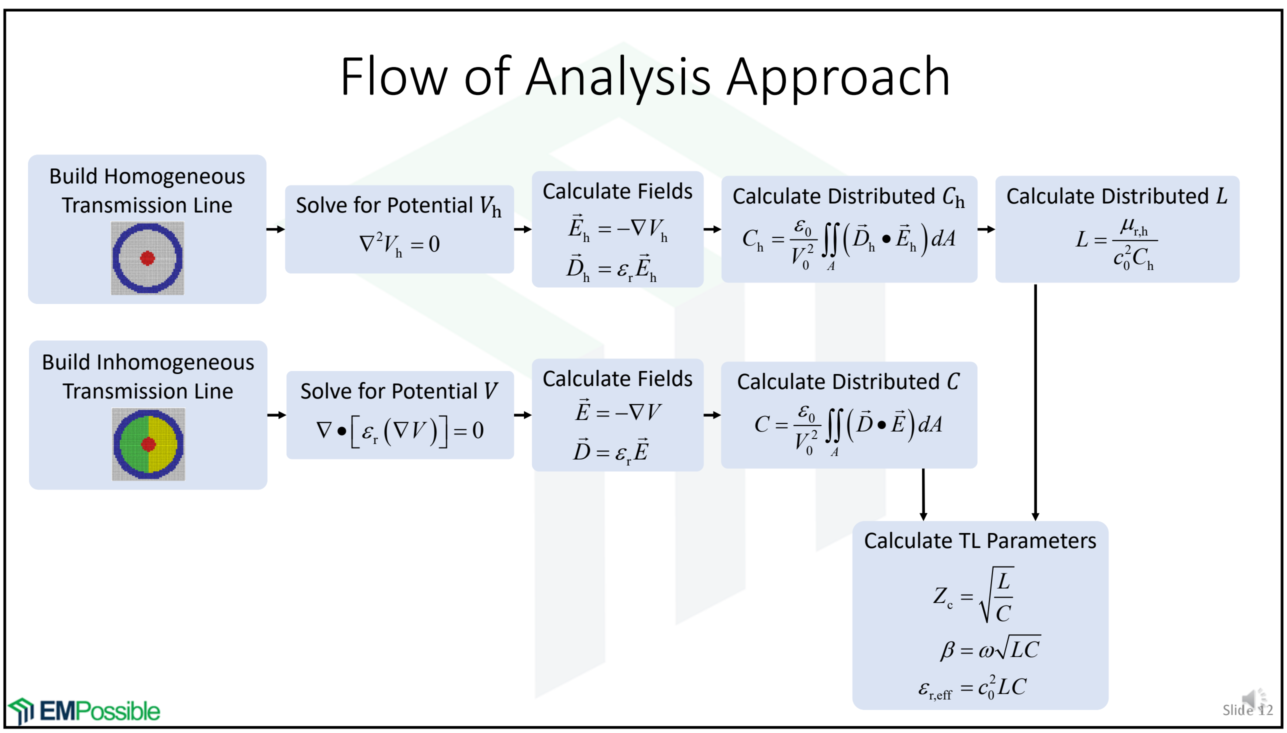
The effective dielectric constant $\varepsilon_\mathrm{r,eff}$ is therefore

$$\boxed{\varepsilon_\mathrm{r,eff} = \left(\frac{c_0}{v}\right)^2 = c_0^2 LC}$$

Both Z_c and $\varepsilon_\mathrm{r,eff}$ are needed to analyze transmission line circuits.

EMPossible 10


Two Step Modeling Approach
Step 1 – Homogeneous Case
outer conductor
air
inner conductor
air
distributed inductance L
Step 2 – Inhomogeneous Case
outer conductor
ε_1
ε_2
inner conductor
air
distributed capacitance C
EMPossible
Slide 11


Flow of Analysis Approach
Build Homogeneous Transmission Line
Solve for Potential V_h
$\nabla^2 V_h = 0$
Calculate Fields
$\vec{E}_h = -\nabla V_h$
$\vec{D}_h = \varepsilon_r \vec{E}_h$
Calculate Distributed C_h
$C_h = \frac{\varepsilon_0}{V_0^2} \iint_A \left(\vec{D}_h \bullet \vec{E}_h\right) dA$
Calculate Distributed L
$L = \frac{\mu_{r,h}}{c_0^2 C_h}$
Build Inhomogeneous Transmission Line
Solve for Potential V
$\nabla \bullet \left[\varepsilon_r \left(\nabla V\right)\right] = 0$
Calculate Fields
$\vec{E} = -\nabla V$
$\vec{D} = \varepsilon_r \vec{E}$
Calculate Distributed C
$C = \frac{\varepsilon_0}{V_0^2} \iint_A \left(\vec{D} \bullet \vec{E}\right) dA$
Calculate TL Parameters
$Z_c = \sqrt{\frac{L}{C}}$
$\beta = \omega\sqrt{LC}$
$\varepsilon_{r,eff} = c_0^2 LC$
EMPossible
Slide 12