



Computational Science:  
Computational Methods in Engineering

# Solving One-Dimensional Ordinary Differential Equations



## Outline

- Solving one-dimensional ordinary differential equations
- Example



# Solving One-Dimensional Ordinary Differential Equations

Slide 3

## Boundary Values

Many differential equations give boundary values, or initial conditions, that must be incorporated into the matrix equation in order to obtain a numerical solution.

For example, a problem may be stated as...

$$\frac{d^2 f(x)}{dx^2} + 5 \frac{df(x)}{dx} + 6f(x) = 0 \quad 0 \leq x \leq 2$$

$$f(0) = 2$$

$$f(2) = 0.2$$

Boundary values

Slide 4

## Why are Boundary Values Needed?

To solve the differential equation, it is first converted to matrix form.

$$\frac{d^2}{dx^2} f(x) + 5 \frac{d}{dx} f(x) + 6f(x) = 0$$

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$

$$[D_x^2][f] + 5[D_x][f] + 6[f] = [0]$$

Next, this is rearranged into standard form  $\mathbf{Ax} = \mathbf{b}$ .

$$[A][f] = [0] \quad [A] = [D_x^2] + 5[D_x] + 6[I]$$

This is not solvable.

$$[f] = [A]^{-1}[0] = [0]$$

The boundary values have to be incorporated in order to obtain a nontrivial solution. When this is done, the matrix equation will become

$$[A'][f] = [b]$$

## Incorporating Boundary Values

It is possible to incorporate the boundary values into the matrix operators that we used to build  $[A]$ . However, it is usually easier to incorporate these directly into the final matrix equation.

The steps are:

1. Build the direct matrix equation  $[A][f] = [0]$ .
2. For each row in  $[A]$  that corresponds to a boundary point:
  1. Replace entire row with 0's.
  2. Insert a '1' into the diagonal, or pivot, position.
  3. Place the boundary value in the same row of  $[b]$ .

## Solve the Matrix Equation

This matrix equation is now in the form

$$[A][f]=[b]$$

This is solved for  $[f]$  as follows.

$$[f]=[A]^{-1}[b]$$

For small to moderate size problems, this can be solved directly using LU decomposition, or just backward division in MATLAB.

`f = A\b;`      **DO NOT USE** `f = inv(A)*b`

For large problems, iterative methods are preferred, but the conditioning of  $[A]$  becomes important and a solution is not guaranteed.

## Example

## Problem Setup

Solve the following ordinary differential equation.

$$\frac{d^2 f(x)}{dx^2} + 5 \frac{df(x)}{dx} + 6f(x) = 0 \quad 0 \leq x \leq 2$$

$$f(0) = 2 \quad f(2) = 0.2$$

```
% DEFINE BOUNDARY VALUES
xa = 0;
xb = 2;
fa = 2;
fb = 0.2;
```

## Formulation of the Matrix $[A]$

Step 1 – Formulate the matrix equation  $[A][f] = [0]$ .

$$\frac{d^2}{dx^2} f(x) + 5 \frac{d}{dx} f(x) + 6f(x) = 0$$

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$

$$[D_x^2][f] + 5[D_x][f] + 6[f] = [0]$$

$$([D_x^2] + 5[D_x] + 6[I])[f] = [0]$$

$$[A][f] = [0]$$

$$[A] = [D_x^2] + 5[D_x] + 6[I]$$

## Calculate the Grid Parameters

Step 2 – Calculate grid parameters  $N$  and  $\Delta x$ .

Choose  $N = 11$ . This is just a first guess!

Calculate  $\Delta x$ .

$$\Delta x = \frac{b-a}{N-1} = \frac{2-0}{11-1} = 0.2$$

```
% GRID PARAMETERS
Nx = 11;
dx = (xb - xa) / (Nx - 1);
```

## Build Matrix Operators

Step 3 – Build Matrix Operators  $[D_x^2]$ ,  $[D_x]$ ,  $[I]$ .

$$[D_x^2] = \begin{bmatrix} -50 & 25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 25 & -50 & 25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 25 & -50 & 25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 25 & -50 & 25 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 25 & -50 & 25 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 25 & -50 & 25 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 25 & -50 & 25 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 25 & -50 & 25 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 25 & -50 & 25 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 25 & -50 & 25 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 25 & -50 \end{bmatrix}$$

$$[I] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[D_x] = \begin{bmatrix} 0 & 2.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2.5 & 0 & 2.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2.5 & 0 & 2.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2.5 & 0 & 2.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2.5 & 0 & 2.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2.5 & 0 & 2.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2.5 & 0 & 2.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2.5 & 0 & 2.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2.5 & 0 & 2.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2.5 & 0 & 2.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2.5 & 0 \end{bmatrix}$$

```
% BUILD MATRIX OPERATORS
I = speye(Nx, Nx);
[DX, DX2] = fdder1d(Nx, dx);
```

## Build Initial Matrix Equation $[A][f] = [0]$

Step 4 – Calculate Initial  $[A]$  and  $[b]$ .

$$[A] = [D_x^2] + 5[D_x] + 6[I] = \begin{bmatrix} -44 & 37.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 12.5 & -44 & 37.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 12.5 & -44 & 37.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 12.5 & -44 & 37.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 12.5 & -44 & 37.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 12.5 & -44 & 37.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 12.5 & -44 & 37.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 12.5 & -44 & 37.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12.5 & -44 & 37.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12.5 & -44 & 37.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12.5 & -44 \end{bmatrix} \quad [b] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

```
% CALCULATE [A] AND [b]
```

```
A = DX2 + 5*DX + 6*I;
```

```
b = sparse(Nx,1);
```

## Incorporate Boundary Values Into $[A]$ and $[b]$

Step 5 – Incorporate Boundary Values into  $[A]$  and  $[b]$ .

$$[A] = [D_x^2] + 5[D_x] + 6[I] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 12.5 & -44 & 37.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 12.5 & -44 & 37.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 12.5 & -44 & 37.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 12.5 & -44 & 37.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 12.5 & -44 & 37.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 12.5 & -44 & 37.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 12.5 & -44 & 37.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12.5 & -44 & 37.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12.5 & -44 & 37.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad [b] = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.2 \end{bmatrix}$$

```
% INCORPORATE BOUNDARY VALUES
```

```
A([1 Nx], :) = 0;
```

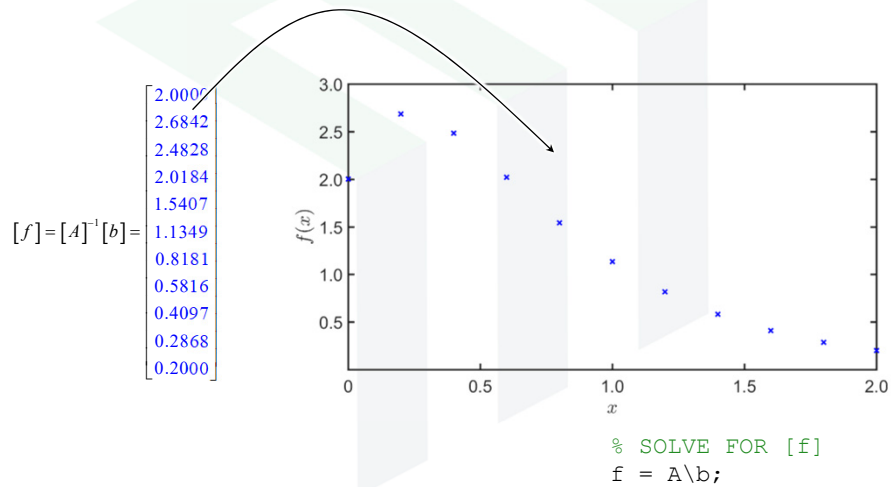
```
A(1,1) = 1;
```

```
A(Nx,Nx) = 1;
```

```
b([1 Nx]) = [ fa fb ];
```

## Solve for Unknown Function $[f]$

Step 6 – Solve for  $[f]$ .

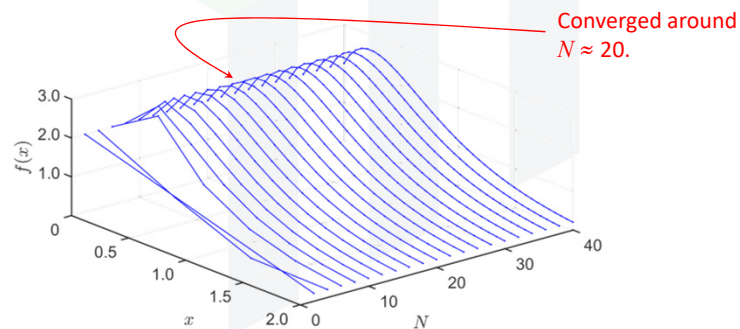


## Check for Convergence

Step 7 – Check for Convergence

A solution was obtained, but it was all based on a guess for how many points  $N$  to use on the grid.

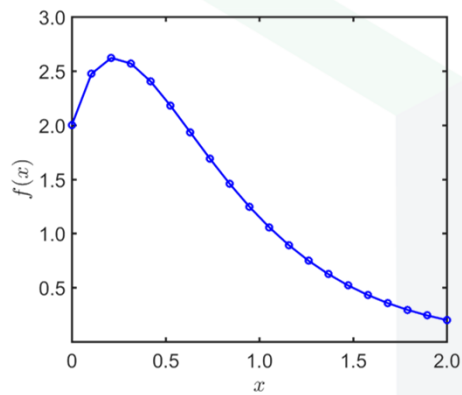
It is critical to check for convergence by increasing the value of  $N$  until the changes in the solution are negligible.





## Obtain Final Converged Answer

Step 8 – Obtain the final answer at convergence (i.e.  $N = 20$ ).



```
% PLOT RESULT
h = plot(x,f,'-ob','LineWidth',2);
h2 = get(h,'Parent');
set(h2,'LineWidth',2,'FontSize',18);
xlabel('$x$','Interpreter','LaTeX');
ylabel('$f(x)$','Interpreter','LaTeX');
T = [0 0.5 1 1.5 2];
L = {'0' '0.5' '1.0' '1.5' '2.0'};
set(gca,'XTick',T,'XTickLabel',L);
T = [0.5 1 1.5 2 2.5 3];
L = {'0.5' '1.0' '1.5' '2.0' '2.5' '3.0'};
set(gca,'YTick',T,'YTickLabel',L);
```

## Analyze the Answer

Step 9 – Post process the data.

The finite-difference method is complete.

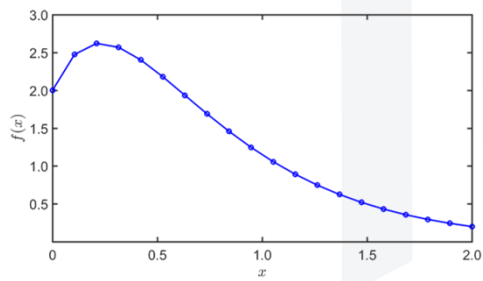
Usually, after obtaining a solution, the results are analyzed further.

## Code Altogether

$$\frac{d^2 f(x)}{dx^2} + 5 \frac{df(x)}{dx} + 6f(x) = 0$$

$$0 \leq x \leq 2$$

$$f(0) = 2 \quad f(2) = 0.2$$



```

% DEFINE BOUNDARY VALUES
xa = 0;
xb = 2;
fa = 2;
fb = 0.2;

% GRID PARAMETERS
Nx = 20;
dx = (xb - xa)/(Nx - 1);

% BUILD MATRIX OPERATORS
I = speye(Nx,Nx);
[DX,DX2] = fdde1d(Nx,dx);

% CALCULATE [A] AND [b]
A = DX2 + 5*DX + 6*I;
b = sparse(Nx,1);

% INCORPORATE BOUNDARY VALUES
A([1 Nx],:) = 0;
A(1,1) = 1;
A(Nx,Nx) = 1;
b([1 Nx]) = [ fa fb ];

% SOLVE PROBLEM
f = A\b;

```