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Computational Science:
Computational Methods in Engineering

Solving One-Dimensional Ordinary Differential Equations

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Outline

- Solving one-dimensional ordinary differential equations
- Example

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Solving One-Dimensional Ordinary Differential Equations

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Boundary Values

Many differential equations give boundary values, or initial conditions, that must be incorporated into the matrix equation in order to obtain a numerical solution.

For example, a problem may be stated as…

$$\frac{d^2 f(x)}{dx^2} + 5\frac{df(x)}{dx} + 6f(x) = 0 \qquad 0 \le x \le 2$$

$$f(0) = 2 \qquad f(2) = 0.2$$

Boundary values

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Why are Boundary Values Needed?

To solve the differential equation, it is first converted to matrix form.

$$\frac{d^2}{dx^2} f(x) + 5\frac{d}{dx} f(x) + 6f(x) = 0$$

$$\downarrow$$

$$\left[D_x^2\right][f] + 5[D_x][f] + 6[f] = [0]$$

Next, this is rearranged into standard form $[A][f] = [0]$.

$$[A][f] = [0] \qquad [A] = \left[D_x^2\right] + 5[D_x] + 6[I]$$

This is not solvable.

$$[f] = [A]^{-1}[0] = [0]$$

The boundary values must be incorporated in order to obtain a nontrivial solution. When this is done, the matrix equation becomes

$$[A'][f] = [b]$$

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Incorporating Boundary Values

The boundary values will be incorporated into the matrix equation $[A][f] = [0]$.

The steps are:

1. Build the matrix equation $[A][f] = [0]$.
2. For each row in $[A]$ that corresponds to a boundary point:
   1. Replace entire row with 0's.
   2. Insert a '1' into the diagonal, or pivot, position.
   3. Place the boundary value in the same row of $[b]$.

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Solve the Matrix Equation

The matrix equation is now in the form

$$[A'][f]=[b]$$

This is solved for $[f]$ as follows.

$$[f]=[A']^{-1}[b]$$

For small to moderate size problems, this can be solved directly using LU decomposition, or just backward division in MATLAB.

`f = A\b;` DO NOT USE `f = inv(A)*b`

For large problems, iterative methods are preferred, but the conditioning of $[A]$ becomes important and a solution is not guaranteed.

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Example

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Problem Setup

Solve the following ordinary differential equation.

$$\frac{d^2 f(x)}{dx^2} + 5\frac{df(x)}{dx} + 6f(x) = 0 \qquad 0 \le x \le 2$$

$$f(0) = 2 \qquad f(2) = 0.2$$

```
% DEFINE BOUNDARY VALUES
xa = 0;
xb = 2;
fa = 2;
fb = 0.2;
```

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Formulation of the Matrix $[A]$

Step 1 – Formulate the matrix equation $[A][f] = [0]$.

$$\frac{d^2}{dx^2} f(x) + 5\frac{d}{dx} f(x) + 6f(x) = 0$$

$$\left[D_x^2\right][f] + 5[D_x][f] + 6[f] = [0]$$

$$\left(\left[D_x^2\right] + 5[D_x] + 6[I]\right)[f] = [0]$$

$$[A][f] = [0]$$

$$[A] = \left[D_x^2\right] + 5[D_x] + 6[I]$$

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Calculate the Grid Parameters

Step 2 – Calculate grid parameters N and Δx.

Choose $N = 11$. This is just a preliminary guess!

Calculate Δx.

$$\Delta x = \frac{b-a}{N-1} = \frac{2-0}{11-1} = 0.2$$

```
% GRID PARAMETERS
Nx = 11;
dx = (xb - xa)/(Nx - 1);
```

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Build Matrix Operators

Step 3 – Build the matrix operators $[D_x^2]$, $[D_x]$, and $[I]$.

$$[D_x^2] = \begin{bmatrix} -50 & 25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 25 & -50 & 25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 25 & -50 & 25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 25 & -50 & 25 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 25 & -50 & 25 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 25 & -50 & 25 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 25 & -50 & 25 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 25 & -50 & 25 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 25 & -50 & 25 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 25 & -50 & 25 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 25 & -50 \end{bmatrix}$$

$$[I] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[D_x] = \begin{bmatrix} 0 & 2.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2.5 & 0 & 2.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2.5 & 0 & 2.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2.5 & 0 & 2.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2.5 & 0 & 2.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2.5 & 0 & 2.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2.5 & 0 & 2.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2.5 & 0 & 2.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2.5 & 0 & 2.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2.5 & 0 & 2.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2.5 & 0 \end{bmatrix}$$

```
% BUILD MATRIX OPERATORS
I = speye(Nx,Nx);
[DX,DX2] = fdder1d(Nx,dx);
```

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Build Initial Matrix Equation $[A][f] = [0]$

Step 4 – Calculate Initial $[A]$ and $[b]$.

$$[A] = \left[D_x^2\right] + 5\left[D_x\right] + 6\left[I\right] = \begin{bmatrix} -44 & 37.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 12.5 & -44 & 37.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 12.5 & -44 & 37.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 12.5 & -44 & 37.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 12.5 & -44 & 37.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 12.5 & -44 & 37.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 12.5 & -44 & 37.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 12.5 & -44 & 37.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12.5 & -44 & 37.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12.5 & -44 & 37.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12.5 & -44 \end{bmatrix} \qquad [b] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

```
% CALCULATE [A] AND [b]
A = DX2 + 5*DX + 6*I;
b = sparse(Nx,1);
```

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Incorporate Boundary Values Into $[A]$ and $[b]$

Step 5 – Incorporate Boundary Values into $[A]$ and $[b]$.

$$[A] = \left[D_x^2\right] + 5\left[D_x\right] + 6\left[I\right] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 12.5 & -44 & 37.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 12.5 & -44 & 37.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 12.5 & -44 & 37.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 12.5 & -44 & 37.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 12.5 & -44 & 37.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 12.5 & -44 & 37.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 12.5 & -44 & 37.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12.5 & -44 & 37.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12.5 & -44 & 37.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad [b] = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.2 \end{bmatrix}$$

```
% INCORPORATE BOUNDARY VALUES
A([1 Nx],:) = 0;
A(1,1)      = 1;
A(Nx,Nx)    = 1;
b([1 Nx])   = [ fa fb ];
```

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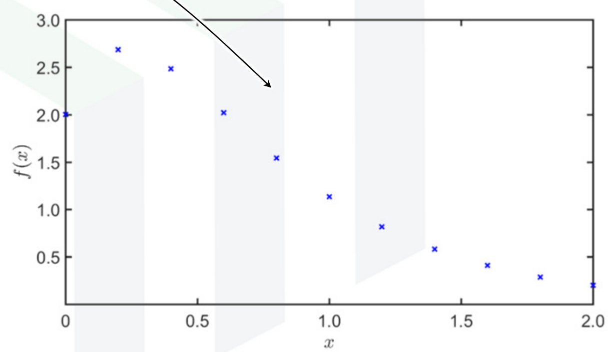
Solve for Unknown Function $[f]$

Step 6 – Solve for $[f]$.

$$[f]=[A]^{-1}[b]=\begin{bmatrix}2.0000\\2.6842\\2.4828\\2.0184\\1.5407\\1.1349\\0.8181\\0.5816\\0.4097\\0.2868\\0.2000\end{bmatrix}$$




```
% SOLVE FOR [f]
f = A\b;
```

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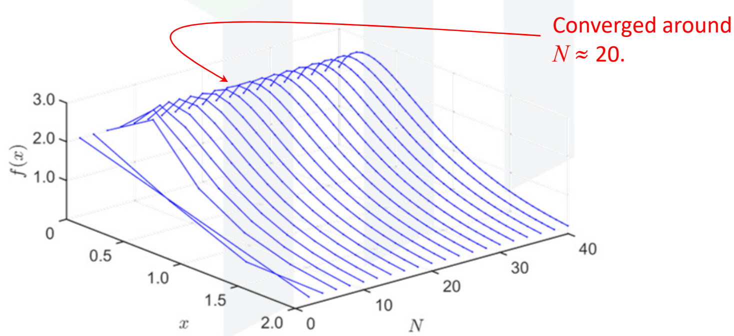
Check for Convergence

Step 7 – Check for Convergence

A solution was obtained, but it was all based on a guess for how many points N to use on the grid.

It is critical to check for convergence by increasing the value of N until the changes in the solution are negligible.



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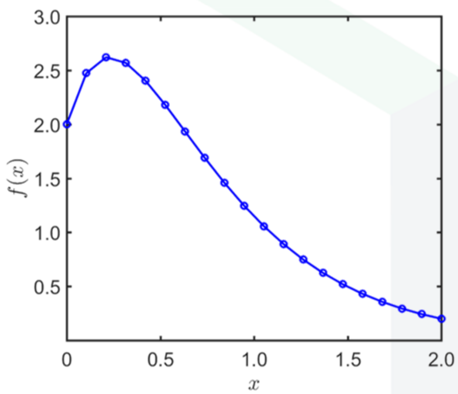
Obtain Final Converged Answer

Step 8 – Obtain the final answer at convergence (i.e. N = 20).




```
% PLOT RESULT
h = plot(x,f,'-ob','LineWidth',2);
h2 = get(h,'Parent');
set(h2,'LineWidth',2,'FontSize',18);
xlabel('$x$','Interpreter','LaTex');
ylabel('$f(x)$','Interpreter','LaTex');
T = [0 0.5 1 1.5 2];
L = {'0' '0.5' '1.0' '1.5' '2.0'};
set(gca,'XTick',T,'XTickLabel',L);
T = [0.5 1 1.5 2 2.5 3];
L = {'0.5' '1.0' '1.5' '2.0' '2.5' '3.0'};
set(gca,'YTick',T,'YTickLabel',L);
```

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Analyze the Answer

Step 9 – Post process the data.

The finite-difference method is complete.

Usually, after obtaining a solution, the results are analyzed further.

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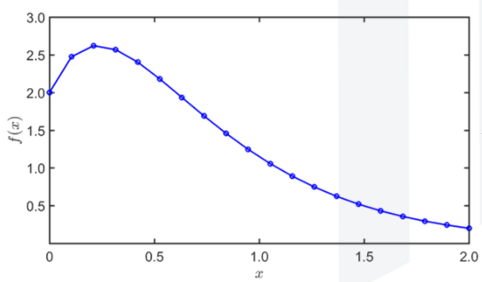
Code Altogether

$$\frac{d^2 f(x)}{dx^2} + 5\frac{df(x)}{dx} + 6f(x) = 0$$

$$0 \le x \le 2$$

$$f(0) = 2 \qquad f(2) = 0.2$$

```
% DEFINE BOUNDARY VALUES
xa = 0;
xb = 2;
fa = 2;
fb = 0.2;

% GRID PARAMETERS
Nx = 20;
dx = (xb - xa)/(Nx - 1);

% BUILD MATRIX OPERATORS
I = speye(Nx,Nx);
[DX,DX2] = fdder1d(Nx,dx);

% CALCULATE [A] AND [b]
A = DX2 + 5*DX + 6*I;
b = sparse(Nx,1);

% INCORPORATE BOUNDARY VALUES
A([1 Nx],:) = 0;
A(1,1)      = 1;
A(Nx,Nx)    = 1;
b([1 Nx])   = [ fa fb ];

% SOLVE PROBLEM
f = A\b;
```



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