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Computational Science:
Computational Methods in Engineering

Solving Systems of Linear Algebraic Equations

Systems of Linear Equations

Very often in science and engineering, problems can be reduced to a system of linear equations.

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}
$$

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\end{aligned}
$$

$a_{ij} \equiv$ constant coefficient (usually known)

$x_i \equiv$ unknown values

$b_i \equiv$ constants (usually excitation)

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Direct Analytical Solution

Suppose it is desired to solve the following system of equations

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3
\end{aligned}
$$

Step 1 – Solve first equation for x_1.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \quad \rightarrow \quad x_1 = \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}}x_2 - \frac{a_{13}}{a_{11}}x_3$$

$$
\begin{aligned}
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3
\end{aligned}
$$

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Direct Analytical Solution

Step 2 – *Forward Substitution* – Substitute this new equation into 2nd and 3rd equations to eliminate x_1.

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\
a'_{22}x_2 + a'_{23}x_3 &= b'_2 \\
a'_{32}x_2 + a'_{33}x_3 &= b'_3
\end{aligned}
$$

$$
a'_{22} = a_{22} - \frac{a_{21}a_{12}}{a_{11}} \qquad a'_{23} = a_{23} - \frac{a_{21}a_{13}}{a_{11}} \qquad b'_2 = b_2 - \frac{a_{21}b_1}{a_{11}}
$$

$$
a'_{32} = a_{32} - \frac{a_{31}a_{12}}{a_{11}} \qquad a'_{33} = a_{33} - \frac{a_{31}a_{13}}{a_{11}} \qquad b'_3 = b_3 - \frac{a_{31}b_1}{a_{11}}
$$

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Direct Analytical Solution

Step 3 – Solve second equation for x_2.

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\
a'_{22}x_2 + a'_{23}x_3 &= b'_2 \quad \rightarrow \quad x_2 = \frac{b'_2}{a'_{22}} - \frac{a'_{23}}{a'_{22}}x_3 \\
a'_{32}x_2 + a'_{33}x_3 &= b'_3
\end{aligned}
$$

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Direct Analytical Solution

Step 4 – *Forward Substitution* – Substitute this new equation into 3rd equation to eliminate x_2.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a'_{22}x_2 + a'_{23}x_3 &= b'_2 \\ a''_{33}x_3 &= b''_3 \end{aligned}$$

$$a''_{33} = a'_{33} - \frac{a'_{32}a'_{23}}{a'_{22}} \qquad b''_3 = b'_3 - \frac{a'_{32}b'_2}{a'_{22}}$$

Observe the triangular form of the equations here.
This will come up again later.
Think of it as an "almost solved" system of equations.

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Direct Analytical Solution

Step 5 – Solve third equation for x_3. Since this is the last equation, the final answer is obtained for x_3.

$$x_3 = \frac{b''_3}{a''_{33}}$$

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Direct Analytical Solution

Step 6 – *Backward Substitution* – Given x_3, calculate x_2 using equation from Step 3.

$$x_2 = \frac{b_2' - a_{23}' x_3}{a_{22}'}$$

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Direct Analytical Solution

Step 7 – *Backward Substitution* – Given x_2 and x_3, calculate x_1 using equation from Step 1.

$$x_1 = \frac{b_1 - a_{12} x_2 - a_{13} x_3}{a_{11}}$$

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