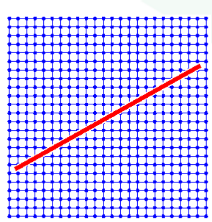
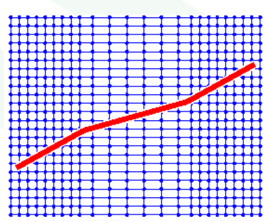
Advanced Electromagnetics:
21st Century Electromagnetics

Stretching Space with Transformation Optics

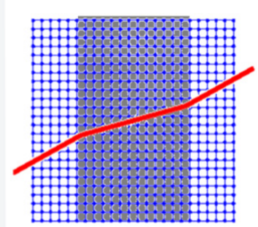
Setup of the Problem



Starting coordinate system



It is desired to “stretch” space in order to move two objects farther apart.



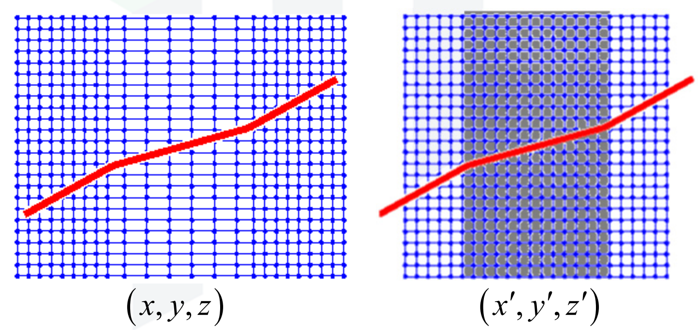
Using transformation optics, calculate metamaterial properties that will do this.

Define the Coordinate Transform

Suppose it is desired to "stretch" the z-axis by a factor of a.

$$x' = x$$
$$y' = y$$
$$z' = z/a$$

It is desired to end in the standard uniform Cartesian coordinate system. This requires starting in a coordinate system that is stretched. Therefore, the coordinate transform must compress space. This will lead to the $[\mu]$ and $[\varepsilon]$ that stretch space.



Calculate the Jacobian

$$x' = x$$
$$y' = y$$
$$z' = z/a$$

$$\Rightarrow \quad [J] = \begin{bmatrix} \partial x'/\partial x & \partial x'/\partial y & \partial x'/\partial z \\ \partial y'/\partial x & \partial y'/\partial y & \partial y'/\partial z \\ \partial z'/\partial x & \partial z'/\partial y & \partial z'/\partial z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/a \end{bmatrix}$$

Calculate Transformed $[\mu']$ and $[\varepsilon']$

$$[\mu'] = \frac{[J][\mu][J]^T}{\det[J]} = \frac{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{bmatrix}\begin{bmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/a \end{bmatrix}^T}{\det\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/a \end{bmatrix}} = \frac{\begin{bmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu/a^2 \end{bmatrix}}{1/a} = \begin{bmatrix} a\mu & 0 & 0 \\ 0 & a\mu & 0 \\ 0 & 0 & \mu/a \end{bmatrix}$$

$$[\varepsilon'] = \frac{[J][\varepsilon][J]^T}{\det[J]} = \frac{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{bmatrix}\begin{bmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/a \end{bmatrix}^T}{\det\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/a \end{bmatrix}} = \frac{\begin{bmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon/a^2 \end{bmatrix}}{1/a} = \begin{bmatrix} a\varepsilon & 0 & 0 \\ 0 & a\varepsilon & 0 \\ 0 & 0 & \varepsilon/a \end{bmatrix}$$

EMPossible

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What Does the Answer Mean?

$$[\mu'] = \begin{bmatrix} a\mu & 0 & 0 \\ 0 & a\mu & 0 \\ 0 & 0 & \mu/a \end{bmatrix} \qquad [\varepsilon'] = \begin{bmatrix} a\varepsilon & 0 & 0 \\ 0 & a\varepsilon & 0 \\ 0 & 0 & \varepsilon/a \end{bmatrix}$$

These tensors have two equal elements and a different third element. → Uniaxial

For $a > 0$, the third element is smaller in value than the first two. → Negative uniaxial

EMPossible

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Space-Stretching Metamaterial

Recall from *Subwavelength Gratings* lecture, a negative uniaxial metamaterial is realized by an array of sheets with alternating material properties.

$$[\varepsilon'] = \begin{bmatrix} \varepsilon_o & 0 & 0 \\ 0 & \varepsilon_o & 0 \\ 0 & 0 & \varepsilon_e \end{bmatrix}$$

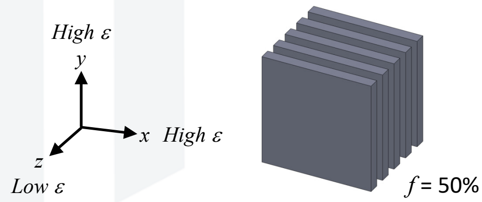
$$\varepsilon_o = a\varepsilon$$

$$\varepsilon_e = \varepsilon / a$$

Stretching factor given tensor:

$$a = \sqrt{\varepsilon_o / \varepsilon_e}$$

$$\varepsilon_{eff} = \sqrt{\varepsilon_o \varepsilon_e}$$



$\varepsilon_e < \varepsilon_o$ Negative Uniaxial

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