Computational Science:
Computational Methods in Engineering

The Newton-Raphson Method

Outline

• Description of the Method
• Notes on Implementation
• Example
Description of the Method

Start with Any Function

\[ f(x) \]
Make an Initial Guess for the Position of the Root \(x_1\)

Evaluate the Function at \(x_1\)
Calculate the Equation of the Line Tangential to the Point on $f(x)$

$y - y_1 = m(x - x_1)$

$f(x) - f(x_1) = f'(x_1)(x - x_1)$

Calculate Where the Line Crosses the $x$-Axis

$0 - f(x_1) = f'(x_1)(x_2 - x_1)$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$
Evaluate the Function at $x_2$

$$f(x)$$

Calculate the Equation of the Line Tangential to the Point on $f(x)$

$$y - y_2 = m(x - x_2)$$
$$f(x) - f(x_2) = f'(x_2)(x - x_2)$$
Calculate Where the Line Crosses the $x$-Axis

Evaluate the Function at $x_3$
And so on…

Converged in five iterations.

Notes on Implementation
Derive analytical expressions for 
\[ f(x) \quad \text{and} \quad f'(x) \] 
or 
\[ \frac{f(x)}{f'(x)} \]

Determine initial guess \( x_1 \)

Calculate \( \Delta x_i \)
\[ \Delta x_i = \frac{f(x_i)}{f'(x_i)} \]

Update root estimate
\[ x_{i+1} = x_i - \Delta x_i \]

Is \( |\Delta x_i| < \text{tol} \)?

Finished!

Poor or Unstable Convergence

Handling Multiple Roots

If \( f(x) \) has multiple roots, perform root finding on the auxiliary function \( u(x) \) instead.

\[
\begin{align*}
  x_{i+1} &= x_i - \frac{f(x_i)}{f'(x_i)} &\Rightarrow& & x_{i+1} &= x_i - \frac{u(x_i)}{u'(x_i)}
\end{align*}
\]

The new update equation can be written completely in terms of \( f(x) \) and its derivatives as follows.

\[
x_{i+1} = x_i - \frac{f(x_i) f'(x_i)}{\left[f'(x_i)\right]^2 - f(x_i) f''(x_i)}
\]

Notes on Newton-Raphson Method

• Does not require bounds.
• Requires a “good” initial guess.
• Requires \( f(x) \) and \( f'(x) \) to be analytical.
• Converges extremely fast for functions that are near linear.
• Algorithm vulnerable to instability
• Can converge to the wrong root if multiple roots exist.
• Newton-Raphson Method ≠ Newton’s Method
  • While closely related, NRM is for root finding whereas NM is for optimization.
Example #1

Let $f(x) = \sin x$. What is the root of $f(x)$ in the proximity of $x = 4$?

Step 1 -- Derive analytical expression for $f(x)/f'(x)$

$$f(x) = \sin x$$

$$\Delta x = \frac{f(x)}{f'(x)} = \frac{\sin x}{\cos x} = \tan x$$

This means the update equation is

$$x_{i+1} = x_i - \Delta x_i = x_i - \tan x_i$$

Step 2 -- MATLAB code

```matlab
xr = 4;
tol = 1e-6;
dx = inf;
while abs(dx) > tol
    dx = tan(xr);
    xr = xr - dx;
end
```

Converges to 3.1416 after 4 iterations.