



Computational Science:  
Computational Methods in Engineering

# Trapezoidal Integration



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## Outline

- Trapezoidal Integration
- Formulation
- Discrete Vs. Trapezoidal Integration



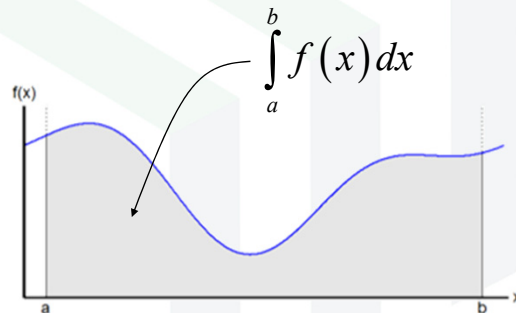
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# Trapezoidal Integration

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## Problem Setup

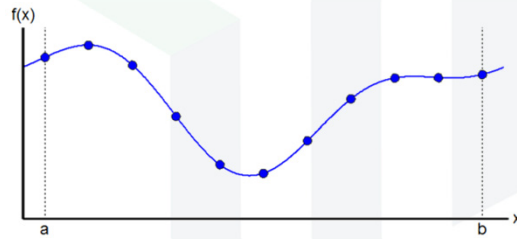
Suppose a function  $f(x)$  is to be integrated from  $a$  to  $b$ .



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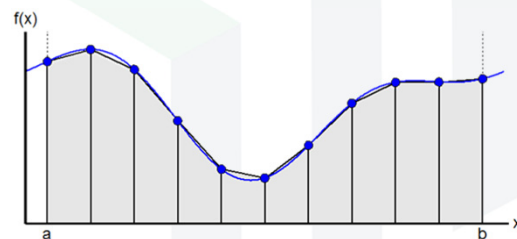
## Solution (1 of 2)

A more accurate technique for numerical integration uses the trapezoidal rule. For this, the points  $x_n$  are placed differently. **Points are placed at the extreme ends and distributed evenly.**



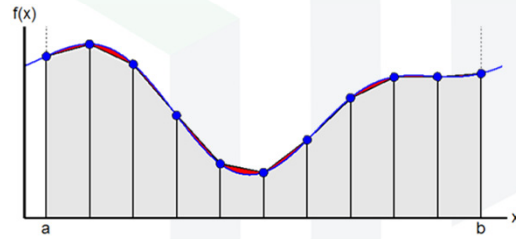
## Solution (2 of 2)

Instead of fitting rectangles under the curve, trapezoids are used. This conforms more closely to the curve.



## Error

Approximating the integral this way still produces some **error**. There is noticeably less error than with discrete integration.

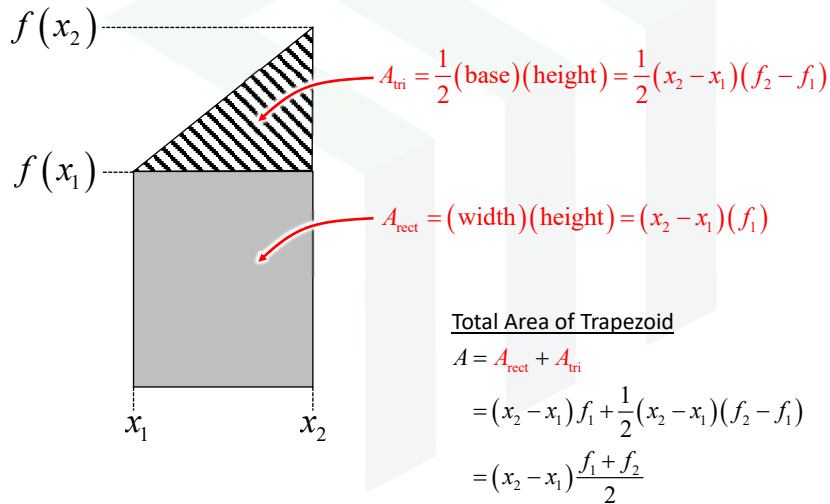


The error for trapezoidal integration is  $E_t(x) = -\frac{1}{12} f''(x)(b-a)^3$

## Formulation

## Formulation (1 of 2)

The total area of a trapezoid is



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## Formulation (2 of 2)

In trapezoidal integration, the definite integral is approximated by adding the areas of all the individual trapezoids.

$$\int_a^b f(x) dx \approx \begin{cases} \sum_{n=1}^N (x_{n+1} - x_n) \frac{f_n + f_{n+1}}{2} & \text{nonuniform spacing} \\ \frac{\Delta x}{2} \sum_{n=1}^N (f_n + f_{n+1}) & \text{uniform spacing} \end{cases}$$

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## Discrete Vs. Trapezoidal Integration

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## Uniform Spacing

When the spacing is uniform, trapezoidal integration reduces to

$$\int_a^b f(x) dx \approx \frac{\Delta x}{2} \sum_{n=1}^N (f_n + f_{n+1})$$

To understand this more deeply, expand the summation over four trapezoids.

$$\sum_{n=1}^4 (f_n + f_{n+1}) = (f_1 + f_2) + (f_2 + f_3) + (f_3 + f_4) + (f_4 + f_5)$$

Observe that each point is included twice, except the two endpoints at  $x = a$  and  $x = b$ .

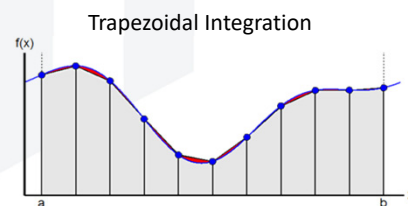
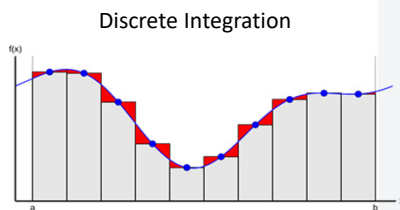
$$\sum_{n=1}^4 (f_n + f_{n+1}) = f_1 + 2f_2 + 2f_3 + 2f_4 + f_5$$

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## Discrete Vs. Trapezoidal Integration (1 of 2)

There are some key differences between discrete and trapezoidal integration:

- Points are distributed differently.
- Discrete integration is easier to implement.
- Trapezoidal integration has less error.
- Trapezoidal more elegantly handles nonuniform spacing.



## Discrete Vs. Trapezoidal Integration (2 of 2)

Compare the equations for both discrete and trapezoidal integration. First, rearrange trapezoidal integration as follows:

$$\int_a^b f(x) dx \approx \frac{\Delta x}{2} \sum_{n=1}^4 (f_n + f_{n+1}) = \Delta x (0.5f_1 + f_2 + f_3 + f_4 + 0.5f_5)$$

The equivalent equation for discrete integration is

$$\int_a^b f(x) dx \approx \Delta x \sum_{n=1}^4 f_n = \Delta x (f_1 + f_2 + f_3 + f_4)$$

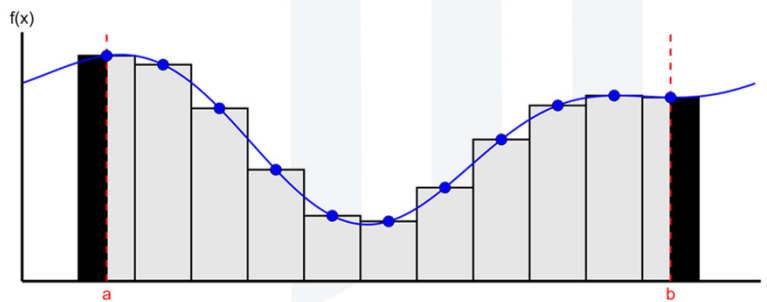
Observe that trapezoidal integration reduces to discrete integration with one extra rectangle added.

## Interpreting Trapezoidal Integration as Discrete Integration

Trapezoidal integration can be written as

$$\int_a^b f(x) dx \approx \frac{\Delta x}{2} \sum_{n=1}^4 (f_n + f_{n+1}) = \Delta x (0.5f_1 + f_2 + f_3 + f_4 + 0.5f_5)$$

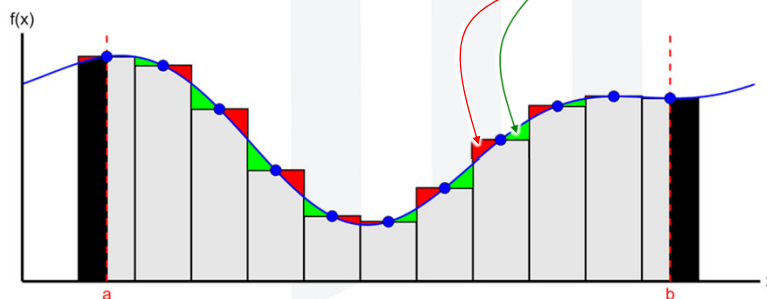
This can be interpreted as a modified discrete integration, but with the half of the first and last rectangles ignored..



## How Can Discrete & Trapezoidal Produce Roughly the Same Error?

- Negative Error
- Positive Error

Positive and negative error tend to cancel within a segment.





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