Computational Science:
Computational Methods in Engineering

Trapezoidal Integration

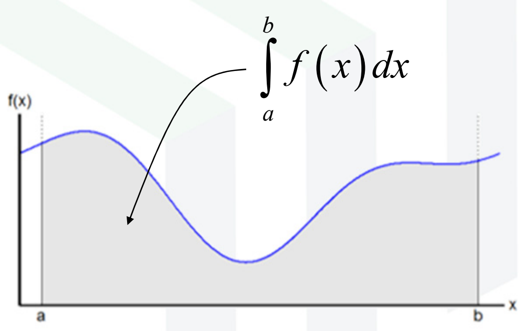
Outline

- Trapezoidal Integration
- Formulation
- Discrete Vs. Trapezoidal Integration

Trapezoidal Integration

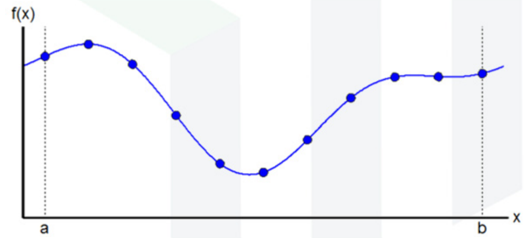
Problem Setup

Suppose a function $f(x)$ is to be integrated from a to b.



Solution (1 of 2)

A more accurate technique for numerical integration uses the trapezoidal rule. For this, the points x_n are placed differently. **Points are placed at the extreme ends and distributed evenly**.



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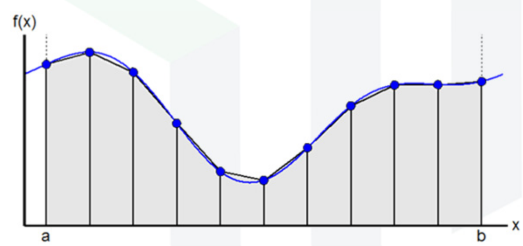
Solution (2 of 2)

Instead of fitting rectangles under the curve, trapezoids are used. This conforms more closely to the curve.



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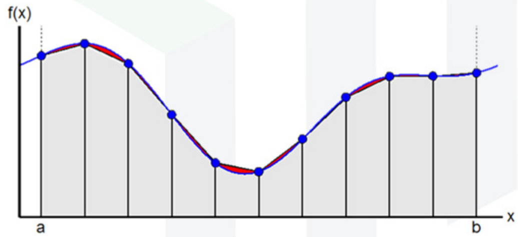
Error

Approximating the integral this way still produces some error. There is noticeably less error than with discrete integration.



The error for trapezoidal integration is $E_t(x) = -\frac{1}{12} f''(x)(b-a)^3$

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Formulation

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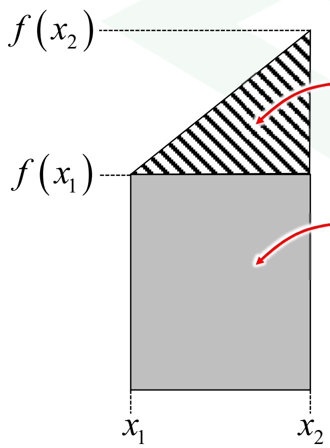
Formulation (1 of 2)

The total area of a trapezoid is



$$A_{\text{tri}} = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(x_2 - x_1)(f_2 - f_1)$$

$$A_{\text{rect}} = (\text{width})(\text{height}) = (x_2 - x_1)(f_1)$$

<u>Total Area of Trapezoid</u>

$$\begin{aligned} A &= A_{\text{rect}} + A_{\text{tri}} \\ &= (x_2 - x_1) f_1 + \frac{1}{2}(x_2 - x_1)(f_2 - f_1) \\ &= (x_2 - x_1)\frac{f_1 + f_2}{2} \end{aligned}$$

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Formulation (2 of 2)

In trapezoidal integration, the definite integral is approximated by adding the areas of all the individual trapezoids.

$$\int_a^b f(x)\,dx \approx \begin{cases} \displaystyle\sum_{n=1}^{N} (x_{n+1} - x_n)\frac{f_n + f_{n+1}}{2} & \text{nonuniform spacing} \\ \displaystyle\frac{\Delta x}{2}\sum_{n=1}^{N} (f_n + f_{n+1}) & \text{uniform spacing} \end{cases}$$

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Discrete Vs. Trapezoidal Integration

Uniform Spacing

When the spacing is uniform, trapezoidal integration reduces to

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\sum_{n=1}^{N}\left(f_n + f_{n+1}\right)$$

To understand this more deeply, expand the summation over four trapezoids.

$$\sum_{n=1}^{4}\left(f_n + f_{n+1}\right) = \left(f_1 + f_2\right) + \left(f_2 + f_3\right) + \left(f_3 + f_4\right) + \left(f_4 + f_5\right)$$

Observe that each point is included twice, except the two endpoints at $x = a$ and $x = b$.

$$\sum_{n=1}^{4}\left(f_n + f_{n+1}\right) = f_1 + 2f_2 + 2f_3 + 2f_4 + f_5$$

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Discrete Vs. Trapezoidal Integration (1 of 2)

There are some key differences between discrete and trapezoidal integration:

- Points are distributed differently.
- Discrete integration is easier to implement.
- Trapezoidal integration has less error.
- Trapezoidal more elegantly handles nonuniform spacing.

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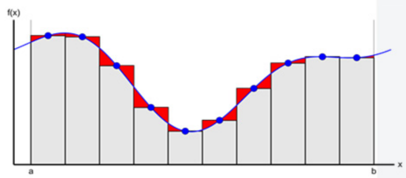
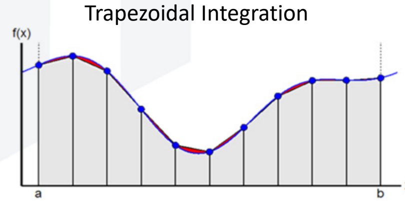
Discrete Integration



Trapezoidal Integration



Discrete Vs. Trapezoidal Integration (2 of 2)

Compare the equations for both discrete and trapezoidal integration. First, rearrange trapezoidal integration as follows:

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\sum_{n=1}^{4}\left(f_n + f_{n+1}\right) = \Delta x\left(0.5 f_1 + f_2 + f_3 + f_4 + 0.5 f_5\right)$$

The equivalent equation for discrete integration is

$$\int_a^b f(x)\,dx \approx \Delta x \sum_{n=1}^{4} f_n = \Delta x\left(f_1 + f_2 + f_3 + f_4\right)$$

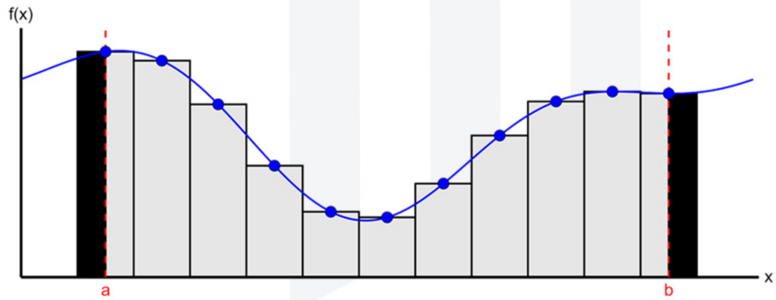
Observe that trapezoidal integration reduces to discrete integration but with one extra rectangle added.

Interpreting Trapezoidal Integration as Discrete Integration

Trapezoidal integration can be written as

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\sum_{n=1}^{4}\left(f_n + f_{n+1}\right) = \Delta x\left(0.5 f_1 + f_2 + f_3 + f_4 + 0.5 f_5\right)$$

This can be interpreted as a modified discrete integration.



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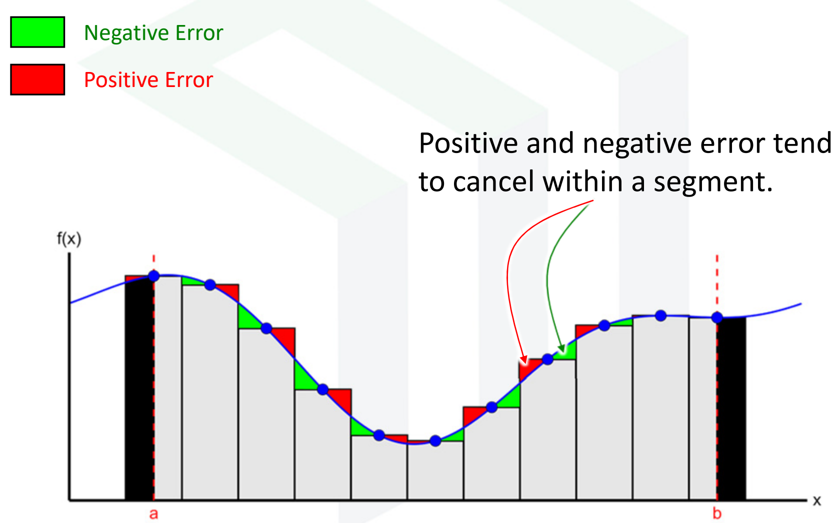
How Can Discrete & Trapezoidal Produce Roughly the Same Error?



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