Electromagnetics:
Microwave Engineering

Rectangular Waveguide Cavity Resonators

Lecture Outline

• Resonant Frequencies of Rectangular Cavity
• Design of a Rectangular Cavity Resonator
Resonant Frequencies of Rectangular Cavity

The transverse electric fields $E_x, E_y$ of the TE$_{mn}$ and TM$_{mn}$ rectangular waveguide mode are

$$\vec{E}_t(x, y, z) = \vec{e}(x, y)(A^+ e^{-i\beta_{mn}z} + A^- e^{i\beta_{mn}z})$$

The propagation constant of the $m, n$th TE or TM mode is

$$\beta_{mn} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

$$k = \omega\sqrt{\mu\varepsilon}$$
Resonant Frequencies of Rectangular Cavity

Applying the conditions for perfectly conducting walls at $z = 0$

$$E_t = 0$$

$$A^+ = -A^-$$

and at $z = d$,

$$\bar{E}_t(x, y, d) = -\bar{e}(x, y)A^+2j\sin(\beta_{mn}d) = 0$$

And the only nontrivial solution is when

$$\beta_{mn}d = l\pi, \quad l = 1, 2, 3 ...$$

the cavity must be an integer multiple of half-wavelength guide at resonant frequency.

Resonant Frequencies of Rectangular Cavity

A resonance wavenumber for the rectangular cavity can be defined as

$$k_{mnl} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{l\pi}{d}\right)^2}$$

We can refer to the $TE_{mnl}$ or $TM_{mnl}$ resonant mode of the cavity, where the standing wave pattern variations correspond to

- $m \rightarrow x$ – direction
- $n \rightarrow y$ – direction
- $l \rightarrow z$ – direction
Resonant Frequencies of Rectangular Cavity

The resonant frequency is given by

\[ f_{mnl} = \frac{ck_{mnl}}{2\pi\sqrt{\mu_r\varepsilon_r}} = \frac{c}{2\pi\sqrt{\mu_r\varepsilon_r}} \sqrt{\left(\frac{mn}{a}\right)^2 + \left(\frac{np}{b}\right)^2 + \left(\frac{lp}{d}\right)^2} \]

For \( b < a < d \), the dominant resonant mode is the \( \text{TE}_{101} \) mode. The dominant \( TM \) resonant mode is the \( \text{TM}_{110} \) mode.

Unloaded \( Q \) of the \( \text{TE}_{10l} \) mode

The unloaded \( Q \) with lossy dielectric filling but perfectly conductor walls is

\[ Q_d = \frac{2\omega W_e}{P_d} = \frac{\varepsilon'}{\varepsilon''} = \frac{1}{\tan \delta} \]

The unloaded \( Q \) with lossy conducting walls but lossless dielectric is

\[ Q_c = \frac{2\omega W_e}{P_c} = \frac{(kad)^3\eta}{2\pi^2 R_s} \frac{1}{(2l^2 a^3 b + 2bd^3 + l^2 a^3 d + ad^3)} \]

And the total unloaded \( Q \) is

\[ Q_0 = \left( \frac{1}{Q_c} + \frac{1}{Q_d} \right)^{-1} \]
Design of a Rectangular Cavity Resonator

A rectangular waveguide cavity is made from a piece of copper WR – 187 H-band waveguide, with \( a = 4.755 \) cm and \( b = 2.215 \) cm. The cavity is filled with polyethylene \( (\varepsilon_r = 2.25, \tan \delta = 0.0004) \). If the resonance is to occur at \( f = 5 \) GHz, find the required length \( d \), and the resulting unloaded \( Q \) for the \( l = 1 \) and \( l = 2 \) resonant modes.

Solution:

We first calculate the wavenumber \( k \) at 5 GHz

\[
k = \frac{2\pi f \sqrt{\varepsilon_r}}{c} = 157.08/m
\]

The dominant mode is \( \text{TE}_{101} \) so \( m = 1, n = 0 \). Then we can find the resonance for \( l = 1 \) and 2

\[
d = \frac{l\pi}{\sqrt{k^2 - \left(\frac{\pi}{a}\right)^2}}
\]
Design of a Rectangular Cavity Resonator

Solution:

For $l = 1$, $d = 2.20$ cm

For $l = 2$, $d = 4.40$ cm

Design of a Rectangular Cavity Resonator

Solution:

Now we can calculate the unloaded $Q$. The $Q$ due to conductor loss is given by

$$Q_c = \frac{(kad)^3 b \eta}{2\pi^2 R_s} \frac{1}{(2l^2 a^3 b + 2bd^3 + l^2 a^3 d + ad^3)}$$

Where $\eta = \frac{377}{\sqrt{\varepsilon_r}} = 251.3$ $\Omega$ for polyethylene.

For $l = 1$, $Q_c = 8,403$

For $l = 2$, $Q_c = 11,898$
Design of a Rectangular Cavity Resonator

Solution:

The $Q$ due to dielectric loss is given by

$$Q_d = \frac{1}{\tan \delta} = 2,500$$

For both $l = 1$ and $2$. Then the total unloaded $Qs$ are

For $l = 1$,

$$Q_0 = \left( \frac{1}{8,403} + \frac{1}{2,500} \right)^{-1} = 1927$$

For $l = 2$,

$$Q_0 = \left( \frac{1}{11,898} + \frac{1}{2,500} \right)^{-1} = 2065$$