Advanced Computation:
Computational Electromagnetics

Rigorous Finite-Difference Analysis of Transmission Lines

Rigorous Analysis of the Transmission Line

Negative Sign Convention

These notes assume the negative sign convention.

Wave propagating in the $+z$ direction.

$$\exp(-jkz)$$

Permittivity with loss.

$$\tilde{\varepsilon} = \varepsilon' - j\varepsilon''$$

Calculating Dielectric Constant ε_r from Conductivity σ and Loss Tangent $\tan\delta$

Dielectrics

$$\tilde{\varepsilon}_r = \varepsilon_r (1 - j\tan\delta)$$

Example: ABS @ 5.0 GHz

$$\varepsilon_r = 2.5$$
$$\tan\delta = 0.005$$
$$\tilde{\varepsilon}_r = (2.5)(1 - j\,\mathrm{t}\,0.005) = 2.5 - j0.0125$$

Metals

$$\tilde{\varepsilon}_r = 1 + \frac{\sigma}{j\omega\varepsilon_0}$$

Example: Copper @ 5.0 GHz

$$\sigma = 5.8\times10^{7}\ \mathrm{S/m}$$
$$\tilde{\varepsilon}_r = 1 + \frac{5.8\times10^{7}\ \mathrm{S/m}}{j2\pi\left(5.0\times10^{9}\ \mathrm{s}^{-1}\right)\left(8.854\times10^{-12}\ \mathrm{F/m}\right)}$$
$$= 1 - 2.1\times10^{8}$$

The Eigen-Value Problem

Using the finite-difference method, transmission lines are analyzed using rigorous hybrid mode analysis of waveguides. The eigen-value problem is

$$\mathbf{\Omega}^2 \begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \end{bmatrix} = \tilde{\gamma}^2 \begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \end{bmatrix}$$

$$\mathbf{\Omega}^2 = \mathbf{PQ}$$

$$\mathbf{P} = \begin{bmatrix} \mathbf{D}_{x'}^{e} \boldsymbol{\varepsilon}_{zz}^{-1} \mathbf{D}_{y'}^{h} & -\left(\mathbf{D}_{x'}^{e} \boldsymbol{\varepsilon}_{zz}^{-1} \mathbf{D}_{x'}^{h} + \boldsymbol{\mu}_{yy} \right) \\ \left(\mathbf{D}_{y'}^{e} \boldsymbol{\varepsilon}_{zz}^{-1} \mathbf{D}_{y'}^{h} + \boldsymbol{\mu}_{xx} \right) & -\mathbf{D}_{y'}^{e} \boldsymbol{\varepsilon}_{zz}^{-1} \mathbf{D}_{x'}^{h} \end{bmatrix}$$

$$\mathbf{Q} = \begin{bmatrix} \mathbf{D}_{x'}^{h} \boldsymbol{\mu}_{zz}^{-1} \mathbf{D}_{y'}^{e} & -\left(\mathbf{D}_{x'}^{h} \boldsymbol{\mu}_{zz}^{-1} \mathbf{D}_{x'}^{e} + \boldsymbol{\varepsilon}_{yy} \right) \\ \left(\mathbf{D}_{y'}^{h} \boldsymbol{\mu}_{zz}^{-1} \mathbf{D}_{y'}^{e} + \boldsymbol{\varepsilon}_{xx} \right) & -\mathbf{D}_{y'}^{h} \boldsymbol{\mu}_{zz}^{-1} \mathbf{D}_{x'}^{e} \end{bmatrix}$$

Calculating the Other Field Components

The eigen-vectors contain only the $\mathbf{e}_x$ and $\mathbf{e}_y$. The magnetic field components $\tilde{\mathbf{h}}_x$ and $\tilde{\mathbf{h}}_y$ are calculated as

$$\begin{bmatrix} \tilde{\mathbf{h}}_x \\ \tilde{\mathbf{h}}_y \end{bmatrix} = \frac{1}{\tilde{\gamma}} \mathbf{Q} \begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \end{bmatrix}$$

From here, the z components $\mathbf{e}_z$ and $\tilde{\mathbf{h}}_z$ are calculated according to

$$\mathbf{e}_z = \boldsymbol{\varepsilon}_{zz}^{-1} \left(\mathbf{D}_{x'}^{\mathrm{h}} \tilde{\mathbf{h}}_y - \mathbf{D}_{y'}^{\mathrm{h}} \tilde{\mathbf{h}}_x \right)$$

$$\tilde{\mathbf{h}}_z = \boldsymbol{\mu}_{zz}^{-1} \left(\mathbf{D}_{x'}^{\mathrm{e}} \mathbf{e}_y - \mathbf{D}_{y'}^{\mathrm{e}} \mathbf{e}_x \right)$$

Post-Processing of the Field to Calculate the Transmission Line Parameters

Calculating Voltage on Line

To calculate the voltage across the line, perform a line integration from conductor to conductor.

$$V_0 = \int_a^b \vec{E} \bullet d\vec{\ell}$$

Calculating Current on Line

To calculate the current in the line, perform a closed-contour line integration around one of the conductors.

$$I_0 = \oint_L \vec{H} \bullet d\vec{\ell}$$

Don't forget to denormalize the magnetic field before calculating the current I_0 or the result will have the incorrect amplitude and phase.

$$\vec{H} = \frac{\vec{\tilde{H}}}{-j\eta_0}$$

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Slide 9

Characteristic Impedance Z_0 and Complex Propagation Constant γ

The characteristic impedance Z_0 is simply

$$Z_0 = \frac{V_0}{I_0}$$

The complex propagation constant γ is calculated from the eigen-value as

$$\gamma = k_0 \tilde{\gamma}$$

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Slide 10

Intermediate Parameters X and A

Recall from transmission line theory how the characteristic impedance Z_0 and complex propagation constant γ were calculated from the transmission line parameters R, L, G and C.

$$X = R + j\omega L$$
$$A = G + j\omega C$$
$$Z_0 = \sqrt{\frac{X}{A}}$$
$$\gamma = \sqrt{XA}$$

Given Z_0 and γ, it is possible to calculate X and A by solving the above equations for these variables.

$$X = \gamma Z_0 \qquad A = \gamma / Z_0$$

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Distributed Parameters R, L, G, and C

Given the intermediate parameters X and A, the distributed transmission line parameters are

$$R = \mathrm{Re}[X] = \mathrm{Re}[\gamma Z_0]$$
$$L = \frac{\mathrm{Im}[X]}{\omega} = \frac{\mathrm{Im}[\gamma Z_0]}{\omega}$$
$$G = \mathrm{Re}[A] = \mathrm{Re}\left[\frac{\gamma}{Z_0}\right]$$
$$C = \frac{\mathrm{Im}[A]}{\omega} = \frac{\mathrm{Im}[\gamma / Z_0]}{\omega}$$

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Analysis Steps (1 of 2)

Step 1 – Perform rigorous finite-difference Analysis

Step 2 – Calculate applied voltage V_0

$$V_0 = \int_a^b \vec{E} \bullet d\vec{\ell}$$

Step 3 – Calculate current in line I_0

$$I_0 = \oint_L \vec{H} \bullet d\vec{\ell}$$

Step 4 – Calculate characteristic impedance Z_0

$$Z_0 = \frac{V_0}{I_0}$$

Analysis Steps (2 of 2)

Step 5 – Calculate intermediate parameters

$$X = \gamma Z_0$$

$$A = \frac{\gamma}{Z_0}$$

Step 6 – Calculate distributed parameters

$$R = -\mathrm{Re}[X]$$

$$L = \frac{\mathrm{Im}[X]}{\omega}$$

$$G = -\mathrm{Re}[A]$$

$$C = \frac{\mathrm{Im}[A]}{\omega}$$