Electromagnetics:

Electromagnetic Field Theory

Electric Potential

1

Outline

- Concept of electric potential V
- Potential difference V_{AB}
- Electric potential around a charge

EMPossible

Slide 2

2

Concept of Electric Potential

Vector Calculus Derivation of Electric Potential

Recall from vector calculus that the gradient of a scalar function cannot have any curl.

$$\nabla \times (\nabla V) = 0$$

Recall for electrostatics that the electric field has zero curl.

$$\nabla \times \vec{E} = 0$$

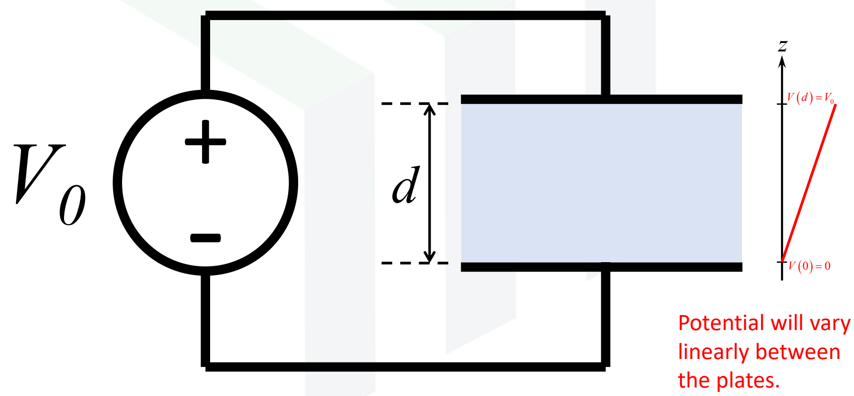
This means that the electric field intensity can be written as the gradient of a scalar function V.

$$\boxed{\vec{E} = -\nabla V}$$

A negative sign is incorporated to enforce the sign convention that electric fields are directed from high to low potential.

Intuitive Derivation of Electric Potential

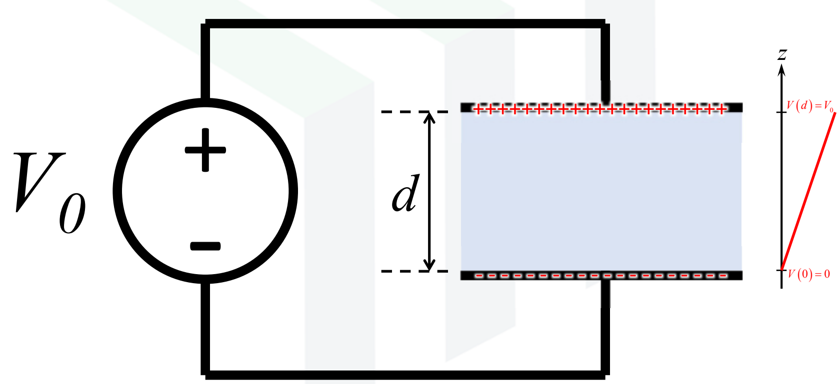
Suppose a voltage V_0 is applied across two plates bounding some medium.



5

Intuitive Derivation of Electric Potential

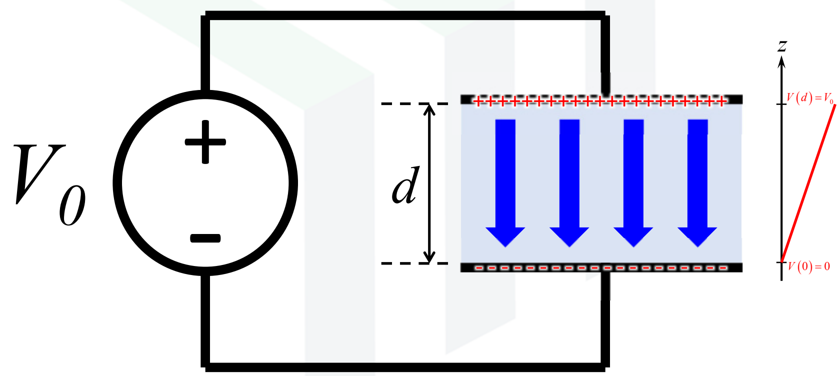
This induces charge in the plates at either side of the medium.



6

Intuitive Derivation of Electric Potential

The charge creates an electric field between the plates that is known to be uniform.



EMPossible

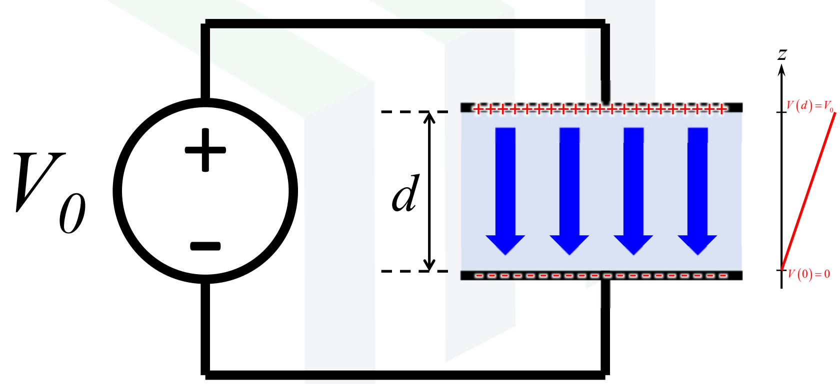
Slide 7

7

Intuitive Derivation of Electric Potential

Conclusion – the electric field is the slope of the voltage.

$$\vec{E} = -\frac{dV}{dz}\hat{a}_z$$



EMPossible

Slide 8

8

Intuitive Derivation of Electric Potential

This can be generalized to 3D using the gradient operation.

$$\vec{E} = -\frac{dV}{dz}\hat{a}_z \quad \Longrightarrow \quad \boxed{\vec{E} = -\nabla V}$$

This is the equation used to calculate the electric field from the electric potential.

EMPossible

Slide 9

9

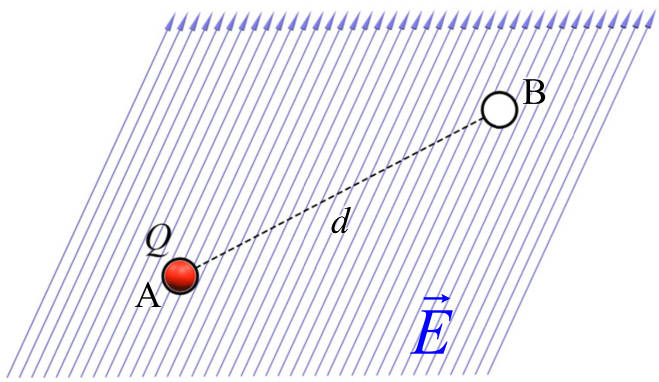
Potential Difference

Slide 10

10

Work Required to Move a Charge

Suppose a point charge Q is moved from point A to point B, a distance of d, in the presence of an electric field $\vec{E}$.



Force on the charge

$$\vec{F} = Q\vec{E}$$

Work done to move charge

$$W = -Fd = -Qd\left|\vec{E}\right|$$

The negative sign indicates the force is external.

This can be generalized to a differential

$$dW = -\vec{F} \bullet d\vec{\ell} = -Q\vec{E} \bullet d\vec{\ell}$$

Integrate to get total work.

$$W = \int_{\mathrm{A}}^{\mathrm{B}} dW = -Q\int_{\mathrm{A}}^{\mathrm{B}} \vec{E} \bullet d\vec{\ell}$$

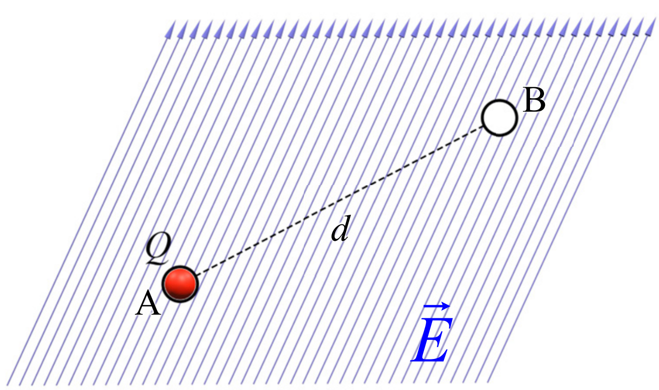
EMPossible

Slide 11

11

Work Required to Move a Charge

Suppose a point charge Q is moved from point A to point B, a distance of d, in the presence of an electric field $\vec{E}$.



Divide total work by Q

$$W = -Q\int_{\mathrm{A}}^{\mathrm{B}} \vec{E} \bullet d\vec{\ell}$$

$$\frac{W}{Q} = -\int_{\mathrm{A}}^{\mathrm{B}} \vec{E} \bullet d\vec{\ell}$$

This is the potential difference between points A and B.

$$V_{\mathrm{AB}} = V_{\mathrm{B}} - V_{\mathrm{A}} = \frac{W}{Q} = -\int_{\mathrm{A}}^{\mathrm{B}} \vec{E} \bullet d\vec{\ell}$$

EMPossible

Slide 12

12

One Application of Potential Difference

The equation below is used to calculate the electric potential from the electric field.

$$V_{AB} = V_B - V_A = -\int_A^B \vec{E} \bullet d\vec{\ell}$$

This is sort of the inverse equation to

$$\vec{E} = -\nabla V$$

where the electric field is calculated from electric potential.

EMPossible Slide 13

13

Notes About Potential Difference

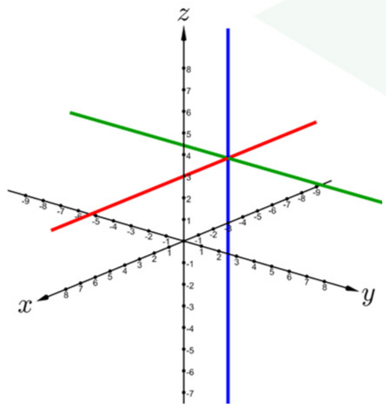
- A is the initial point and B is the final point. This is important for the sign convention.
- $V_{AB} < 0$ indicates a loss in potential energy because work is being done by the field.
- $V_{AB} > 0$ indicates a gain in potential energy because an external agent must be doing the work.
- V_{AB} is independent of the path taken from A to B.
- V_{AB} is measured in joules per Coulomb (J/C), or volts (V).

EMPossible Slide 14

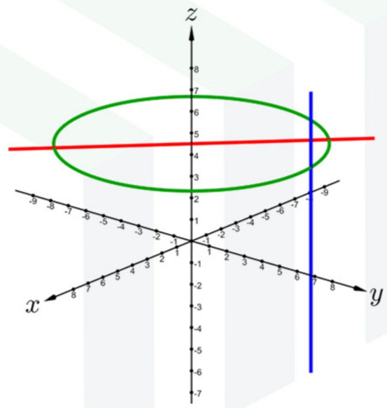
14

Recall Constant Coordinate Lines

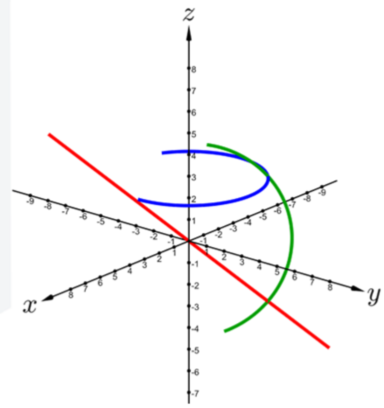
Cartesian (x, y, z)



Cylindrical (ρ, ϕ, z)



Spherical (r, θ, ϕ)



These are the paths typically chosen to calculate line integrals like $\int_L \vec{E} \cdot d\vec{\ell}$

EMPossible

Slide 15

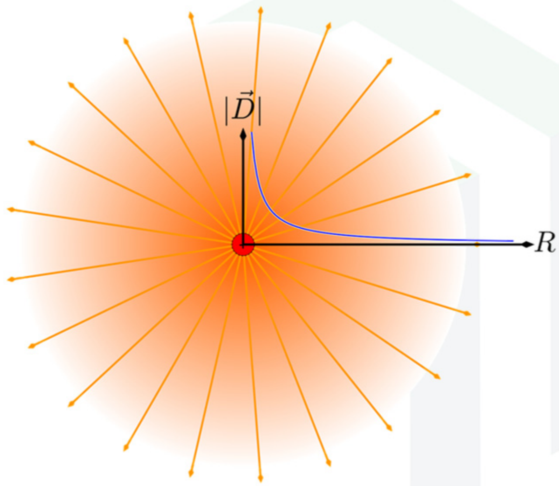
15

Electric Potential Due to Charge

Slide 16

16

Recall the Electric Fields Around a Charge



Electric Flux Density

$$\vec{D} = \frac{Q}{4\pi R^2}\hat{a}_R$$

Electric Field Intensity

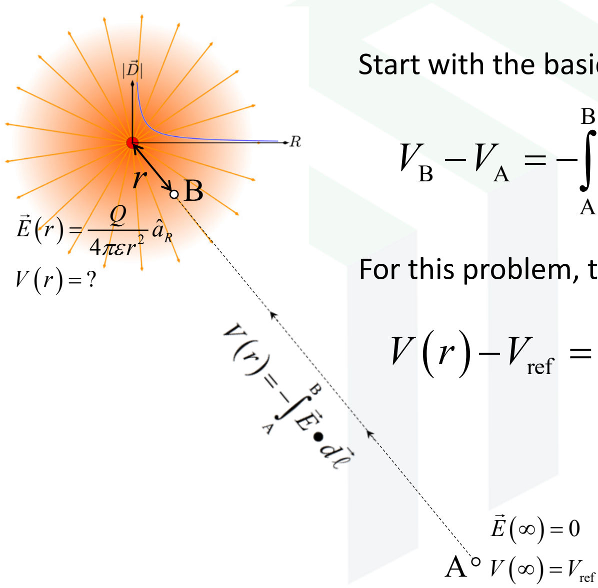
$$\vec{E} = \frac{\vec{D}}{\varepsilon} = \frac{Q}{4\pi\varepsilon R^2}\hat{a}_R$$

EMPossible

Slide 17

17

Derivation Setup



Start with the basic equation to calculate V from $\vec{E}$.

$$V_B - V_A = -\int_A^B \vec{E} \bullet d\vec{\ell}$$

For this problem, the equation becomes

$$V(r) - V_{\text{ref}} = -\int_\infty^r \frac{Q}{4\pi\varepsilon r'^2}\hat{a}_R \bullet dr'\hat{a}_R$$

EMPossible

Slide 18

18

The Derivation

$$V(r) - V_{\text{ref}} = -\int_{\infty}^{r} \frac{Q}{4\pi\varepsilon r'^2} \hat{a}_R \bullet dr'\hat{a}_R$$ Equation from last slide.

$$= -\frac{Q}{4\pi\varepsilon} \int_{\infty}^{r} \frac{1}{r'^2} dr'$$ Bring constants to outside of integral. Perform dot product $\hat{a}_R \bullet dr'\hat{a}_R = dr'$

$$= -\frac{Q}{4\pi\varepsilon}\left(-\frac{1}{r'}\right)\Bigg|_{\infty}^{r}$$ Calculate anti-derivative.

$$= \frac{Q}{4\pi\varepsilon}\left(\frac{1}{r'}\right)\Bigg|_{\infty}^{r}$$ Cancel negative signs.

$$= \frac{Q}{4\pi\varepsilon}\left[\frac{1}{r} - \frac{1}{\infty}\right]$$ Evaluate anti-derivative at limits.

$$V(r) = \frac{Q}{4\pi\varepsilon r} + V_{\text{ref}}$$ Simplify and bring V_{ref} to right side.

EMPossible Slide 19

19

Electric Potential V Around a Point Charge Q

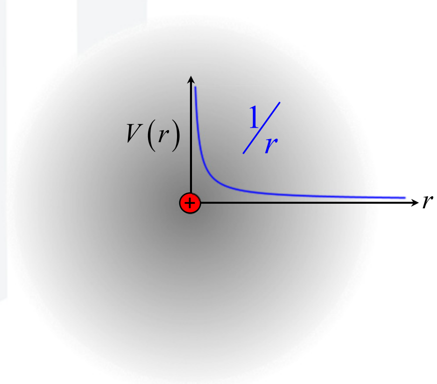
$$V(r) = \frac{Q}{4\pi\varepsilon r} + V_{\text{ref}} \quad \text{or} \quad V(\vec{r}) = \frac{Q}{4\pi\varepsilon\left|\vec{r} - \vec{r}_Q\right|} + V_{\text{ref}}$$

A single potential has almost no meaning.

Only the potential difference between two points is ever of interest.

Any background potential, or reference potential V_{ref}, can be chosen.

Usually, the reference potential V_{ref} is not written explicitly.



EMPossible Slide 20

20

Electric Potential Due to Charge Distributions

Point Charge

Q

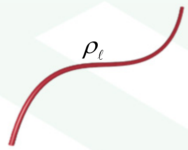
Charge

Q (C)

Electric Potential

$$V = \frac{Q}{4\pi\varepsilon\left|\vec{r} - \vec{r}_Q\right|}$$

Line Charge



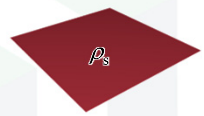
Line Charge Density

ρ_ℓ (C/m)

Electric Potential

$$V = \frac{1}{4\pi\varepsilon}\int_\ell \frac{\rho_\ell}{\left|\vec{r} - \vec{r}'\right|} d\ell$$

Sheet Charge



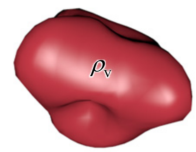
Surface Charge Density

ρ_s $\left(\text{C/m}^2\right)$

Electric Potential

$$V = \frac{1}{4\pi\varepsilon}\iint_S \frac{\rho_s}{\left|\vec{r} - \vec{r}'\right|} ds$$

Volume Charge



Volume Charge Density

ρ_v $\left(\text{C/m}^3\right)$

Electric Potential

$$V = \frac{1}{4\pi\varepsilon}\iiint_v \frac{\rho_v}{\left|\vec{r} - \vec{r}'\right|} dv$$

EMPossible

Slide 21

21