



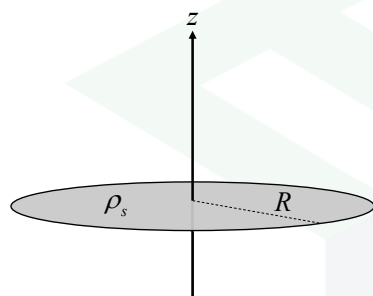
Electromagnetics:
Electromagnetic Field Theory

Example:
Uniform Circular Plate Charge



1

Total Charge



What is the total charge Q_{Total} ?

1. Draw the problem.
2. Choose a coordinate system.
Cylindrical
3. Write general equation.
4. Write expressions for each term.
5. Choose limits of integration.

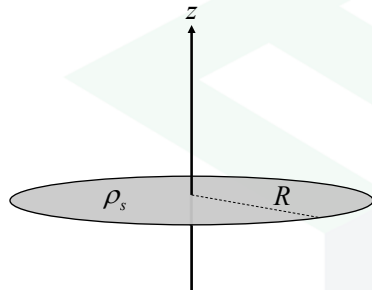
$$Q_{\text{Total}} = \iint_S \rho_s ds$$

$$\rho_s = \rho_s \quad ds = \rho d\rho d\phi$$

$$Q_{\text{Total}} = \int_{\phi=0}^{2\pi} \int_{\rho=0}^R \rho_s \rho d\rho d\phi$$

2

Total Charge



What is the total charge Q_{Total} ?

6. Solve the integral.

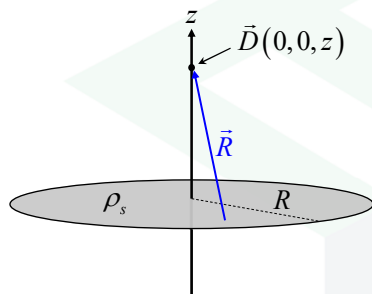
$$\begin{aligned}
 Q_{\text{Total}} &= \int_{\phi=0}^{2\pi} \int_{\rho=0}^R \rho_s \rho d\rho d\phi \\
 &= \rho_s \int_{\phi=0}^{2\pi} \left(\int_{\rho=0}^R \rho d\rho \right) d\phi \\
 &= \rho_s \int_{\phi=0}^{2\pi} \left(\frac{\rho^2}{2} \Big|_{\rho=0}^R \right) d\phi \\
 &= \frac{\rho_s R^2}{2} \int_{\phi=0}^{2\pi} d\phi \\
 &= \frac{\rho_s R^2}{2} \cdot 2\pi \\
 \boxed{Q_{\text{Total}} = \pi R^2 \rho_s}
 \end{aligned}$$

7. Interpret the result.

$$Q_{\text{Total}} = \rho_s S \quad \text{for uniform charge density}$$

3

Total Field



What is the total field \vec{D} ?

1. Draw the problem.

2. Choose a coordinate system.

Cylindrical

3. Write general equation.

$$\vec{D}_{\text{Total}} = \iint_s \frac{\rho_s ds}{4\pi R^2} \hat{a}_R = \frac{\rho_s}{4\pi} \iint_s \frac{\vec{R}}{|\vec{R}|^3} ds$$

4. Write expressions for each term.

$$\rho_s = \rho_s \quad \vec{R} = (0, \phi, z) - (\rho, \phi, 0) = (-\rho, 0, z)$$

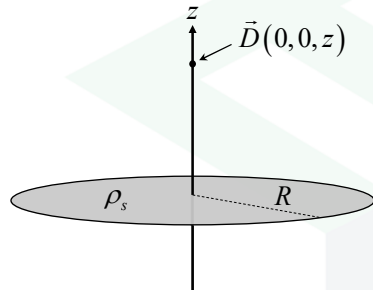
$$ds = \rho d\rho d\phi \quad |\vec{R}| = \sqrt{\rho^2 + z^2}$$

5. Choose limits of integration.

$$\vec{D}_{\text{Total}} = \frac{\rho_s}{4\pi} \int_{\phi=0}^{2\pi} \int_{\rho=0}^R \frac{-\rho \hat{a}_\rho + z \hat{a}_z}{(\rho^2 + z^2)^{3/2}} \rho d\rho d\phi$$

4

Total Field



What is the total field \vec{D} ?

6. Solve the integral.

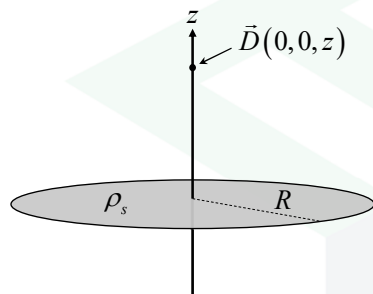
$$\vec{D}_{\text{Total}} = \frac{\rho_s}{4\pi} \int_{\phi=0}^{2\pi} \int_{\rho=0}^R \frac{-\rho \hat{a}_\rho + z \hat{a}_z}{(\rho^2 + z^2)^{3/2}} \rho d\rho d\phi$$

Due to symmetry, we can ignore the ρ component.

$$\begin{aligned} \vec{D}_{\text{Total}} &= \frac{\rho_s}{4\pi} \int_{\phi=0}^{2\pi} \int_{\rho=0}^R \frac{-\cancel{\rho \hat{a}_\rho} + z \hat{a}_z}{(\rho^2 + z^2)^{3/2}} \rho d\rho d\phi \\ &= \frac{\rho_s z}{4\pi} \hat{a}_z \int_{\phi=0}^{2\pi} \int_{\rho=0}^R (\rho^2 + z^2)^{-3/2} \rho d\rho d\phi \\ &= \frac{\rho_s z}{4\pi} \hat{a}_z \int_{\rho=0}^R \left(\int_{\phi=0}^{2\pi} d\phi \right) (\rho^2 + z^2)^{-3/2} \rho d\rho \\ &= \frac{\rho_s z}{4\pi} \hat{a}_z \int_{\rho=0}^R (2\pi) (\rho^2 + z^2)^{-3/2} \rho d\rho \\ &= \frac{\rho_s z}{2} \hat{a}_z \int_{\rho=0}^R (\rho^2 + z^2)^{-3/2} \rho d\rho \end{aligned}$$

5

Total Field



What is the total field \vec{D} ?

6. Solve the integral.

$$\vec{D}_{\text{Total}} = \frac{\rho_s z}{2} \hat{a}_z \int_{\rho=0}^R (\rho^2 + z^2)^{-3/2} \rho d\rho$$

Let $v = \rho^2 + z^2$

Then $dv = 2\rho d\rho + \cancel{0}$

$$d\rho = \frac{dv}{2\rho}$$

$$\rho_1 = 0 \rightarrow v_1 = 0^2 + z^2 = z^2$$

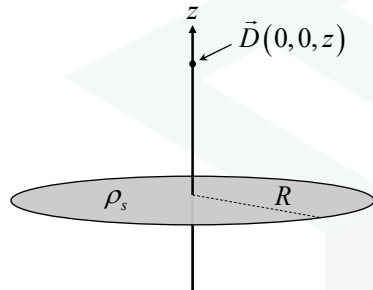
$$\rho_2 = R \rightarrow v_2 = R^2 + z^2$$

Our integral in terms of v is then

$$\begin{aligned} &= \frac{\rho_s z}{2} \hat{a}_z \int_{v=z^2}^{R^2+z^2} v^{-3/2} \rho \frac{dv}{2\rho} \\ &= \frac{\rho_s z}{4} \hat{a}_z \int_{v=z^2}^{R^2+z^2} v^{-3/2} dv \end{aligned}$$

6

Total Field



What is the total field \vec{D} ?

6. Solve the integral.

$$\begin{aligned}
 \vec{D}_{\text{Total}} &= \frac{\rho_s z}{4} \hat{a}_z \int_{v=z^2}^{R^2+z^2} v^{-3/2} dv \\
 &= \frac{\rho_s z}{4} \hat{a}_z \left. \frac{v^{-1/2}}{-1/2} \right|_{z^2}^{R^2+z^2} \\
 &= -\frac{\rho_s z}{2} \hat{a}_z \left. \frac{1}{\sqrt{v}} \right|_{z^2}^{R^2+z^2} \\
 &= -\frac{\rho_s z}{2} \hat{a}_z \left(\frac{1}{\sqrt{R^2+z^2}} - \frac{1}{\sqrt{z^2}} \right) \\
 &= \frac{\rho_s}{2} \left(\frac{z}{\sqrt{z^2}} - \frac{z}{\sqrt{R^2+z^2}} \right) \hat{a}_z
 \end{aligned}$$

$$\boxed{\vec{D}_{\text{Total}} = \frac{\rho_s}{2} \left(1 - \frac{z}{\sqrt{R^2+z^2}} \right) \hat{a}_z}$$