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Electromagnetics:
Electromagnetic Field Theory

Example: Uniform Spherical Charge

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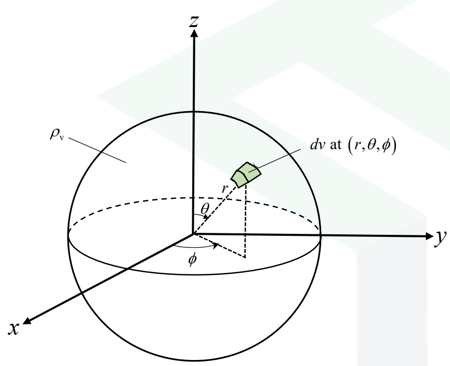
Example #5 – Spherical Volume Charge



What is the total charge Q_{Total}?

1. Draw the problem.
2. Choose a coordinate system.
   Spherical
3. Write general equation.
   $Q_{\text{Total}} = \iiint_{v} \rho_v dv$
4. Write expressions for each term.
   $\rho_v = \rho_v \qquad dv = r^2 \sin\theta dr d\theta d\phi$
5. Choose limits of integration.
   $Q_{\text{Total}} = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \int_{r=0}^{R} \rho_v r^2 \sin\theta dr d\theta d\phi$

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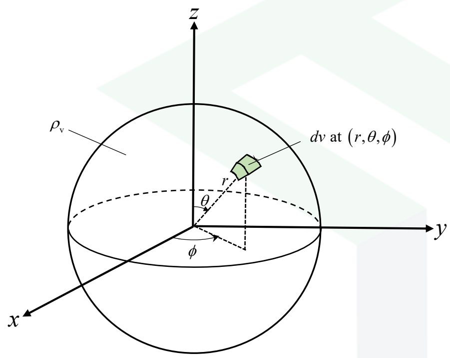
Example #5 – Spherical Volume Charge



What is the total charge Q_{Total}?

6. Solve the integral.

$$Q_{\text{Total}} = \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\int_{r=0}^{R} \rho_v r^2 \sin\theta \, dr\, d\theta\, d\phi$$

$$= \rho_v \int_{\theta=0}^{\pi}\left[\int_{\phi=0}^{2\pi}\left(\int_{r=0}^{R} r^2 dr\right)d\phi\right]\sin\theta\, d\theta$$

$$= \rho_v \int_{\theta=0}^{\pi}\left[\int_{\phi=0}^{2\pi}\left(\left.\frac{r^3}{3}\right|_{r=0}^{R}\right)d\phi\right]\sin\theta\, d\theta$$

$$= \frac{\rho_v R^3}{3}\int_{\theta=0}^{\pi}\left[\int_{\phi=0}^{2\pi} d\phi\right]\sin\theta\, d\theta$$

$$= \frac{\rho_v R^3}{3}\int_{\theta=0}^{\pi}\left[\left.\phi\right|_{\phi=0}^{2\pi}\right]\sin\theta\, d\theta$$

$$= \frac{\rho_v R^3}{3}\int_{\theta=0}^{\pi}\left[2\pi\right]\sin\theta\, d\theta$$

$$= \frac{2\pi\rho_v R^3}{3}\int_{\theta=0}^{\pi}\sin\theta\, d\theta$$

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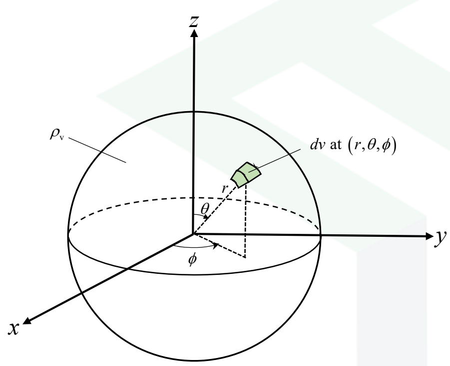

What is the total charge Q_{Total}?

6. Solve the integral.

$$Q_{\text{Total}} = \frac{2\pi\rho_v R^3}{3}\int_{\theta=0}^{\pi}\sin\theta\, d\theta$$

$$= \frac{2\pi\rho_v R^3}{3}\left.\left(-\cos\theta\right)\right|_{\theta=0}^{\pi}$$

$$= \frac{2\pi\rho_v R^3}{3}\left[\left(-\cos\pi\right)-\left(-\cos 0\right)\right]$$

$$= \frac{2\pi\rho_v R^3}{3}\left[1+1\right]$$

$$\boxed{Q_{\text{Total}} = \rho_v \frac{4\pi R^3}{3}}$$

7. Interpret the result.

$Q_{\text{Total}} = \rho_v V$ *for uniform charge density*

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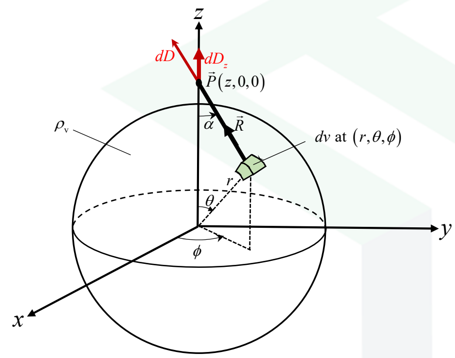


What is the total field $\vec{D}_{\text{Total}}$?

1. Draw the problem.
2. Choose a coordinate system.

   Spherical

3. Write general equation.

   $\vec{D}_{\text{Total}} = \iiint_{v} \frac{\rho_v dv}{4\pi R^2} \hat{a}_R$

4. Write expressions for each term.

   Due to symmetry, the field will only point outward in the $\hat{a}_r$ direction.

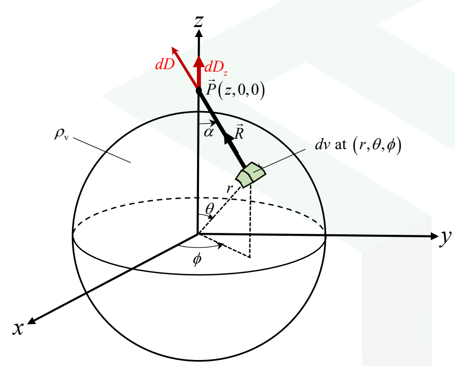
   For this reason, we can solve for the field at any convenient point and we will know the field at any other point.

   We will choose to calculate $\vec{D}_{\text{Total}}$ along the z-axis.

   $\hat{a}_R\big|_z \rightarrow \cos\alpha \hat{a}_r$

   $\vec{D}_{\text{Total}} = \iiint_{v} \frac{\rho_v dv}{4\pi R^2} \cos\alpha \hat{a}_r$

Example #5 – Spherical Volume Charge



What is the total field $\vec{D}_{\text{Total}}$?

4. Write expressions for each term (continued)

$$\vec{D}_{\text{Total}} = \iiint_{v} \frac{\rho_v dv}{4\pi R^2} \cos\alpha \hat{a}_r$$

$$dv = r'^2 \sin\theta' dr' d\theta' d\phi'$$

Prime notation for position of integration within sphere.

$$R^2 = z^2 + r'^2 - 2zr'\cos\theta'$$

Apply law of cosines to figure

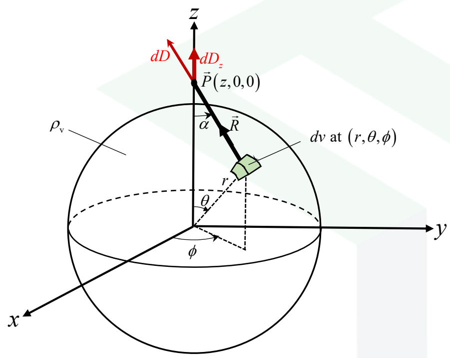
$$\cos\theta' = \frac{z^2 + r'^2 - R^2}{2zr'}$$

$$r'^2 = z^2 + R^2 - 2zR\cos\alpha$$

Apply law of cosines to figure again.

$$\cos\alpha = \frac{z^2 - r'^2 + R^2}{2zR}$$

Example #5 – Spherical Volume Charge



What is the total field $\vec{D}_{\text{Total}}$?

5. Choose limits of integration

It will be easiest to integrate in terms of R and r'.

$dv = r'^2 \sin\theta' dr' d\theta' d\phi'$ Need to convert this

Relate $d\theta'$ and dR by differentiating…

$$\cos\theta' = \frac{z^2 + r'^2 - R^2}{2zr'}$$

$$-\sin\theta' d\theta' = \frac{0 + 0 - 2RdR}{2zr'}$$

$$\sin\theta' d\theta' = \frac{R}{zr'} dR$$

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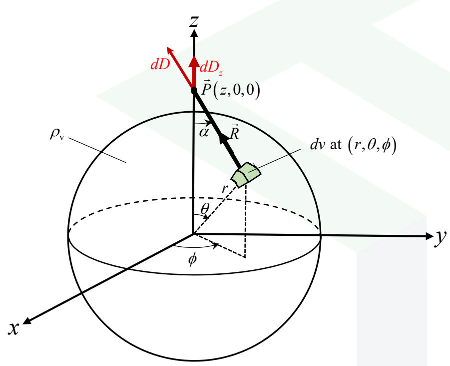
Example #5 – Spherical Volume Charge



What is the total field $\vec{D}_{\text{Total}}$?

5. Choose limits of integration (continued)

$$\vec{D}_{\text{Total}} = \frac{\rho_v}{4\pi} \hat{a}_r \iiint_v \frac{dv}{R^2} \cos\alpha \qquad dv = r'^2 \sin\theta' dr' d\theta' d\phi' \qquad \cos\alpha = \frac{z^2 - r'^2 + R^2}{2zR}$$

$$= \frac{\rho_v}{4\pi} \hat{a}_r \iiint_v \frac{\left(r'^2 \sin\theta' dr' d\theta' d\phi'\right)}{R^2} \left(\frac{z^2 - r'^2 + R^2}{2zR}\right)$$

$$= \frac{\rho_v}{4\pi} \hat{a}_r \iiint_v \frac{z^2 - r'^2 + R^2}{2zR^3} r'^2 \sin\theta' dr' d\theta' d\phi'$$

$$\sin\theta' d\theta' = \frac{R}{zr'} dR$$

$$= \frac{\rho_v}{4\pi} \hat{a}_r \iiint_v \frac{z^2 - r'^2 + R^2}{2zR^3} r'^2 \frac{R}{zr'} dR dr' d\phi'$$

$$= \frac{\rho_v}{4\pi} \hat{a}_r \iiint_v \frac{z^2 - r'^2 + R^2}{2z^2 R^2} r' dR dr' d\phi'$$

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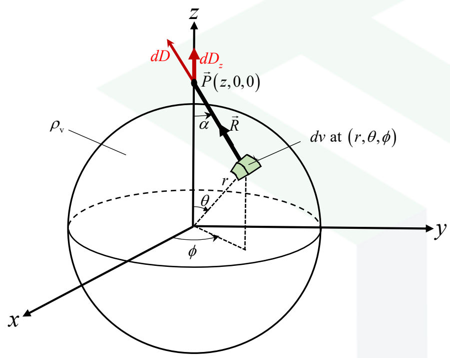
Example #5 – Spherical Volume Charge



What is the total field $\vec{D}_{\text{Total}}$?

5. Choose limits of integration (continued)

$$\vec{D}_{\text{Total}} = \frac{\rho_v}{4\pi}\hat{a}_r \int_{\phi'=0}^{2\pi}\int_{r'=0}^{a}\int_{R=z-r'}^{z+r'} \frac{z^2 - r'^2 + R^2}{2z^2 R^2} r'dRdr'd\phi'$$

6. Solve integration

$$= \frac{\rho_v}{8\pi z^2}\hat{a}_r \int_{r'=0}^{a}\int_{R=z-r'}^{z+r'} \underbrace{\left(\int_{\phi'=0}^{2\pi} d\phi'\right)}_{2\pi} \left(\frac{z^2 - r'^2 + R^2}{R^2}\right) r'dRdr'$$

$$= \frac{\rho_v}{4z^2}\hat{a}_r \int_{r'=0}^{a}\int_{R=z-r'}^{z+r'} \left(\frac{z^2 - r'^2}{R^2} + 1\right) r'dRdr'$$

$$= \frac{\rho_v}{4z^2}\hat{a}_r \int_{r'=0}^{a} \left(-\frac{z^2 - r'^2}{R} + R\right)\Bigg|_{z-r'}^{z+r'} r'dr'$$

$$\left(-\frac{z^2 - r'^2}{R} + R\right)\Bigg|_{z-r'}^{z+r'} = 4r'$$

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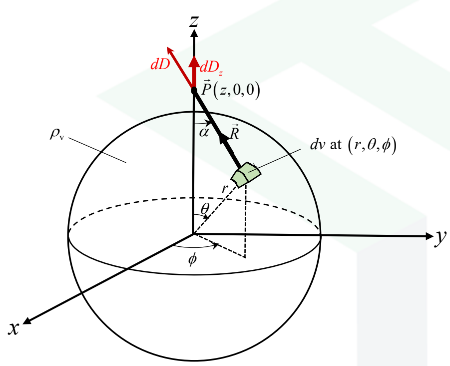
Example #5 – Spherical Volume Charge



What is the total field $\vec{D}_{\text{Total}}$?

6. Solve integration (continued)

$$\vec{D}_{\text{Total}} = \frac{\rho_v}{4z^2}\hat{a}_r \int_{r'=0}^{a} (4r')r'dr' = \frac{\rho_v}{z^2}\hat{a}_r \int_{r'=0}^{a} r'^2 dr'$$

$$= \frac{\rho_v}{z^2}\hat{a}_r \frac{r'^3}{3}\Bigg|_0^a = \frac{\rho_v}{3z^2}\hat{a}_r\, r'^3\Big|_0^a = \frac{\rho_v}{3z^2}\hat{a}_r \left(a^3 - 0^3\right)$$

$$\boxed{\vec{D}_{\text{Total}} = \frac{\rho_v a^3}{3z^2}\hat{a}_r = \frac{\rho_v a^3}{3r^2}\hat{a}_r}$$

7. Interpret the result

Recall $Q_{\text{Total}} = \rho_v \frac{4\pi R^3}{3}$

$$\vec{D}_{\text{Total}} = \frac{Q_{\text{Total}}}{4\pi r^2}\hat{a}_r$$

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