Electromagnetics:
Electromagnetic Field Theory

Energy in Electrostatic Fields

1

Derivation (1 of 5)

It was previously found that energy in a volume charge is

$$W = \frac{1}{2}\iiint_{\mathrm{v}} \rho_{\mathrm{v}} V dv$$

Recall from Maxwell's equations that $\rho_{\mathrm{v}} = \nabla \cdot \vec{D}$.

$$W = \frac{1}{2}\iiint_{\mathrm{v}} \left(\nabla \bullet \vec{D}\right) V dv$$

Recall the product rule for divergence $\nabla \cdot \left(f\vec{A}\right) = f\left(\nabla \cdot \vec{A}\right) + \vec{A} \cdot \nabla f$

$$\nabla \bullet \left(V\vec{D}\right) = V\left(\nabla \bullet \vec{D}\right) + \vec{D} \bullet \nabla V$$

$$\left(\nabla \bullet \vec{D}\right) V = \nabla \bullet \left(V\vec{D}\right) - \vec{D} \bullet \nabla V$$

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Derivation (2 of 5)

Apply the product rule for our equation for work.

$$W = \frac{1}{2}\iiint_{v}\left(\nabla \bullet \vec{D}\right)V dv$$

$$= \frac{1}{2}\iiint_{v}\left[\nabla \bullet \left(V\vec{D}\right) - \vec{D} \bullet \nabla V\right] dv$$

$$= \frac{1}{2}\iiint_{v}\left[\nabla \bullet \left(V\vec{D}\right)\right] dv - \frac{1}{2}\iiint_{v}\left[\vec{D} \bullet \nabla V\right] dv$$

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Derivation (3 of 5)

Recall the divergence theorem

$$\oiint_{S} \vec{F} \bullet d\vec{s} = \iiint_{V}\left(\nabla \bullet \vec{F}\right) dv$$

Apply this to the equation for work.

$$W = \underbrace{\frac{1}{2}\iiint_{v}\left[\nabla \bullet \left(V\vec{D}\right)\right] dv}_{\frac{1}{2}\oiint_{S}\left(V\vec{D}\right)\bullet d\vec{s}} - \frac{1}{2}\iiint_{v}\left[\vec{D} \bullet \nabla V\right] dv$$

$$W = \frac{1}{2}\oiint_{S}\left(V\vec{D}\right)\bullet d\vec{s} - \frac{1}{2}\iiint_{v}\left[\vec{D} \bullet \nabla V\right] dv$$

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Derivation (4 of 5)

Look more closely at the surface integral.

$$W = \frac{1}{2}\oiint_S \left(V\vec{D}\right)\bullet d\vec{s} - \frac{1}{2}\iiint_V \left[\vec{D}\bullet\nabla V\right]dv$$

$V \propto \frac{1}{r}$ $\left|\vec{D}\right| \propto \frac{1}{r^2}$ $\left|d\vec{s}\right| \propto r^2$

$$\text{Overall} \propto \frac{1}{r}\frac{1}{r^2}r^2 = \frac{1}{r}$$

Any surface S can be chosen.

As the surface is enlarged out to infinity, the surface integral becomes negligible relative to the volume integral.

$$W = \cancel{\frac{1}{2}\oiint_S \left(V\vec{D}\right)\bullet d\vec{s}} - \frac{1}{2}\iiint_V \left[\vec{D}\bullet\nabla V\right]dv$$

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Derivation (5 of 5)

The equation for work is now

$$W = -\frac{1}{2}\iiint_V \left[\vec{D}\bullet\nabla V\right]dv$$

Associate the negative sign with ∇V.

$$W = \frac{1}{2}\iiint_V \left[\vec{D}\bullet\left(-\nabla V\right)\right]dv$$

This is the electric field intensity $\vec{E}$.

$$\boxed{W = \frac{1}{2}\iiint_V \left(\vec{D}\bullet\vec{E}\right)dv}$$

This is the general equation for energy stored in the electrostatic field.

It is valid for anisotropic and inhomogeneous media.

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Electrostatic Energy in LHI Media

The more common expression for energy in the electrostatic field is for the special case of linear, homogeneous, and isotropic (LHI) media.

In isotropic media there is $\vec{D} = \varepsilon\vec{E}$.

$$W = \frac{1}{2}\iiint_{v}\left(\vec{D}\bullet\vec{E}\right)dv$$

$$= \frac{1}{2}\iiint_{v}\left(\varepsilon\vec{E}\bullet\vec{E}\right)dv$$

$$\boxed{W = \frac{1}{2}\iiint_{v}\varepsilon\left|\vec{E}\right|^{2}dv}$$

Simpler equation that is only valid in LHI media.

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Electrostatic Energy Density

Total energy has been calculated by integrating.

$$W = \iiint_{v}\left(\frac{1}{2}\vec{D}\bullet\vec{E}\right)dv \qquad W = \iiint_{v}\left(\frac{1}{2}\varepsilon\left|\vec{E}\right|^{2}\right)dv$$

These expressions must be energy density w.

Instead, think of calculating total energy by integrating the energy density w.

$$W = \iiint_{v}w\,dv \qquad w = \begin{cases}\frac{1}{2}\vec{D}\bullet\vec{E} & \text{General case}\\ \frac{1}{2}\varepsilon\left|\vec{E}\right|^{2} & \text{LHI media}\end{cases}$$

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