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Electromagnetics:
Electromagnetic Field Theory

Electrostatic Boundary Conditions

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Outline

- Introduction to boundary conditions
- Boundary conditions for dielectric-dielectric interface
- Boundary conditions for dielectric-conductor interface

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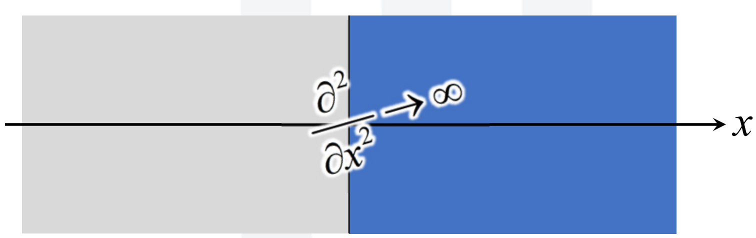
Introduction to Boundary Conditions

What Are Boundary Conditions?

Electromagnetic problems are most often solved using differential equations.

$$\frac{d^2E}{dz^2} + k^2E = 0 \qquad \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

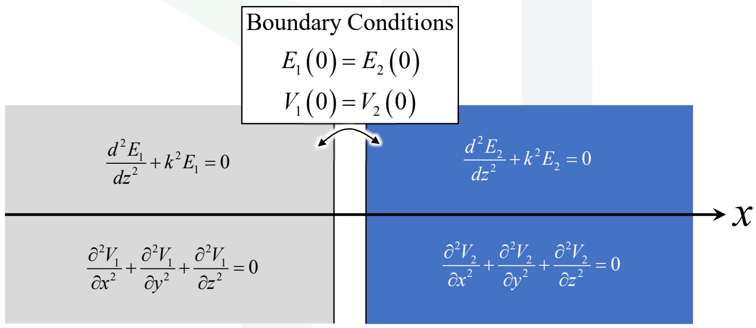
The problem with derivatives is that they are infinite at discontinuities.



What Are Boundary Conditions?

Differential equations must be solved in each homogeneous region separately.

…and then connect the solutions via boundary conditions.



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Deriving Boundary Conditions

Integral equations do not require boundary conditions as long as they do not contain derivatives.

For this reason, the electromagnetic boundary conditions will be derived using Maxwell's equations in integral form.

$$0 = \oint_L \vec{E} \bullet d\vec{\ell} \quad \Longrightarrow$$

Boundary conditions for tangential component of electric fields.

$$Q = \oiint_S \vec{D} \bullet d\vec{s} \quad \Longrightarrow$$

Boundary conditions for normal component of electric fields.

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Boundary Conditions for Dielectric-Dielectric Interface

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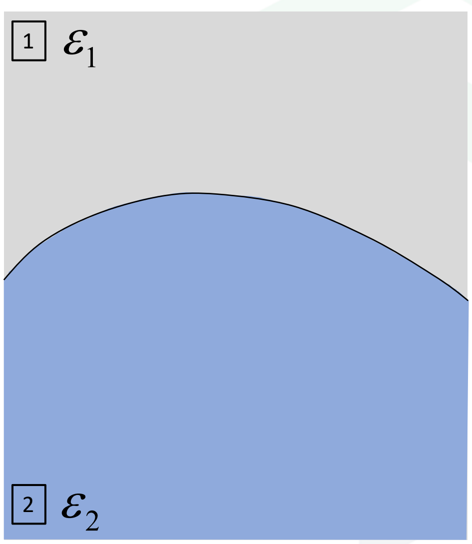
Analysis Setup



Let's examine the interface between two different dielectrics.

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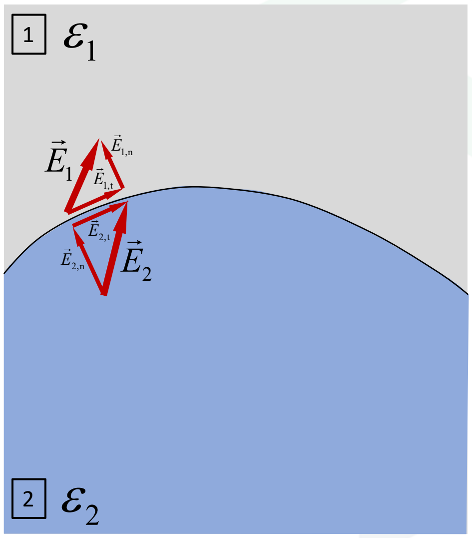
Analysis Setup



Let's examine the interface between two different dielectrics.

Examine the relation between electric fields on either side of the interface, so that if one is known the other can be calculated.

It will be useful to separate the field on either side of the interface into tangential and normal components.

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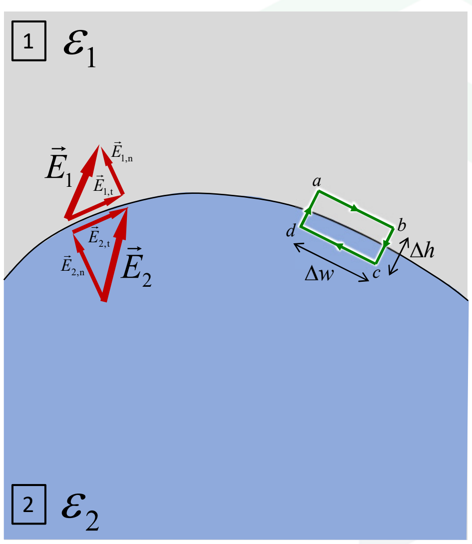
Derivation of Tangential BCs



Apply the following integral to a closed path spanning some section of the interface.

$$
\begin{aligned}
0 &= \oint_L \vec{E} \bullet d\vec{\ell} \\
&= \int_a^b \vec{E} \bullet d\vec{\ell} + \int_b^0 \vec{E} \bullet d\vec{\ell} + \int_0^c \vec{E} \bullet d\vec{\ell} \\
&\quad + \int_c^d \vec{E} \bullet d\vec{\ell} + \int_d^0 \vec{E} \bullet d\vec{\ell} + \int_0^a \vec{E} \bullet d\vec{\ell} \\
&= E_{1,t}\Delta w - E_{1,n}\frac{\Delta h}{2} - E_{2,n}\frac{\Delta h}{2} \\
&\quad - E_{2,t}\Delta w + E_{2,n}\frac{\Delta h}{2} + E_{1,n}\frac{\Delta h}{2}
\end{aligned}
$$

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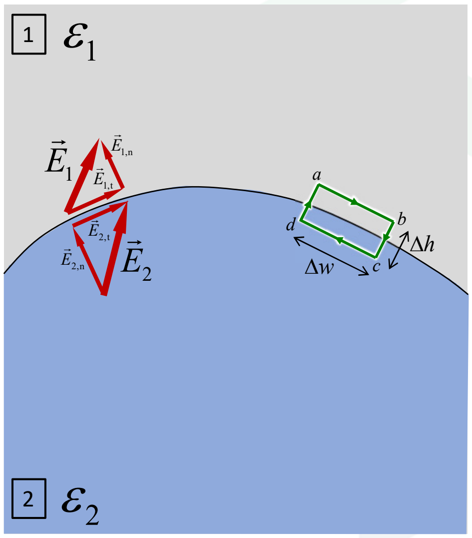
Derivation of Tangential BCs



Cancel like terms with opposite sign.

$$0 = E_{1,t}\Delta w - \cancel{E_{1,n}\frac{\Delta h}{2}} - \cancel{E_{2,n}\frac{\Delta h}{2}}$$

$$-E_{2,t}\Delta w + \cancel{E_{2,n}\frac{\Delta h}{2}} + \cancel{E_{1,n}\frac{\Delta h}{2}}$$

$$= E_{1,t}\Delta w - E_{2,t}\Delta w$$

From this, it is concluded that the tangential component of $\vec{E}$ is continuous across the interface.

$$\boxed{\vec{E}_{1,t} = \vec{E}_{2,t}}$$

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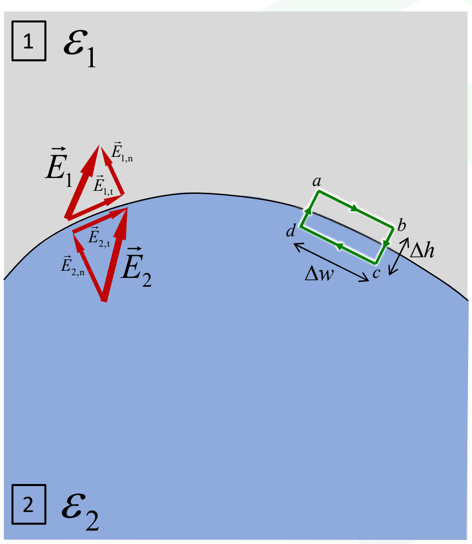
Derivation of Tangential BCs



Apply the constitutive relation to get the boundary condition for $\vec{D}$.

$$\vec{E}_{1,t} = \vec{E}_{2,t}$$

$$\boxed{\frac{\vec{D}_{1,t}}{\varepsilon_1} = \frac{\vec{D}_{2,t}}{\varepsilon_2}}$$

The tangential component of $\vec{D}$ is NOT continuous across the interface, but the ratio of $\vec{D}_t/\varepsilon$ is.

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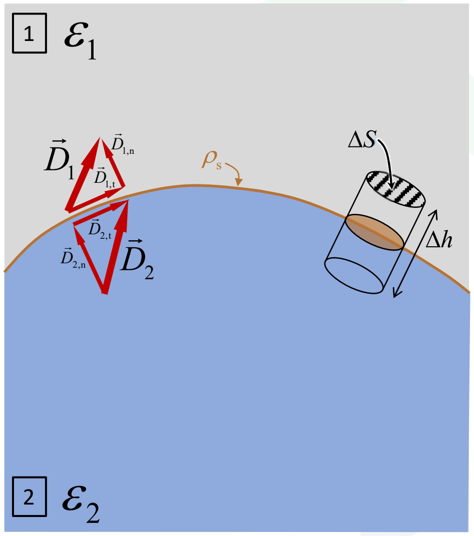
Derivation of Normal BCs



Place some charge density ρ_s on the surface.

Apply the following surface integral to a pillbox spanning the interface.

$$Q = \oiint_S \vec{D} \bullet d\vec{s}$$

Separate the closed-surface integral into three separate surface integrals.

$$Q = \iint_{\text{top}} \vec{D} \bullet d\vec{s} + \iint_{\text{bottom}} \vec{D} \bullet d\vec{s} + \iint_{\text{sides}} \vec{D} \bullet d\vec{s}$$

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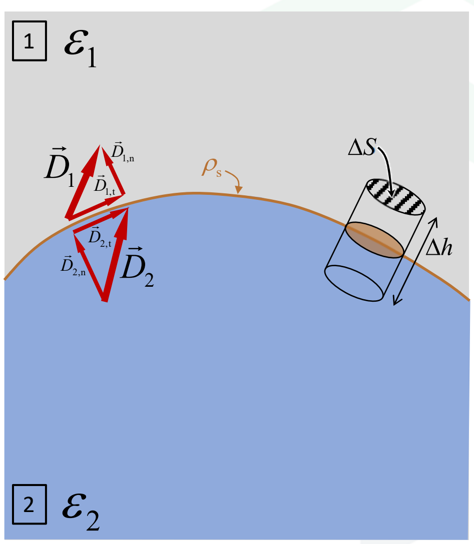
Derivation of Normal BCs



In the limit as $\Delta h \rightarrow 0$

$$Q = \iint_{\text{top}} \vec{D} \bullet d\vec{s} + \iint_{\text{bottom}} \vec{D} \bullet d\vec{s} + \cancel{\iint_{\text{sides}} \vec{D} \bullet d\vec{s}}$$

$$= D_{1,n}\Delta S - D_{2,n}\Delta S$$

The total charge encompassed within the pillbox is

$$Q = \rho_s \Delta S$$

Putting these together gives

$$\rho_s \Delta S = D_{1,n}\Delta S - D_{2,n}\Delta S$$

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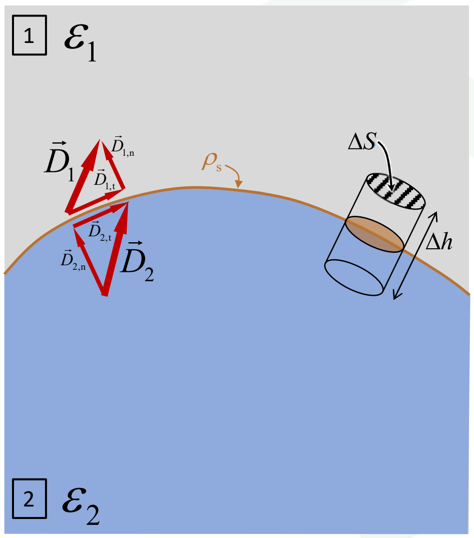
Derivation of Normal BCs



The final boundary condition is then

$$\rho_s \Delta S = D_{1,n} \Delta S - D_{2,n} \Delta S$$

$$\boxed{\vec{D}_{1,n} - \vec{D}_{2,n} = \rho_s}$$

In the absence of charge (i.e. $\rho_s = 0$)

$$\boxed{\vec{D}_{1,n} = \vec{D}_{2,n} \qquad (\rho_s = 0)}$$

The normal component of $\vec{D}$ is continuous across the interface.

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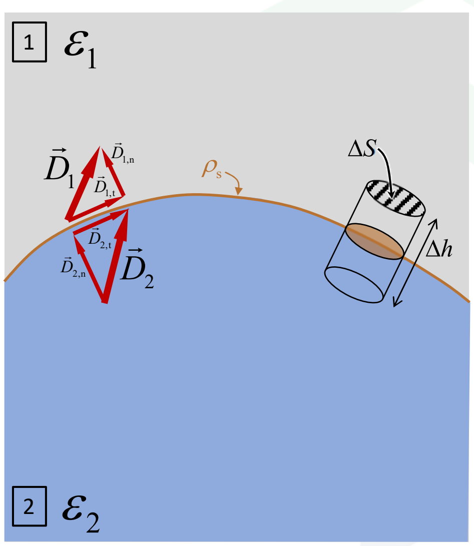
Derivation of Normal BCs



Apply the constitutive relation to get the boundary condition for $\vec{E}$.

$$\vec{D}_{1,n} - \vec{D}_{2,n} = \rho_s$$

$$\boxed{\varepsilon_1 \vec{E}_{1,n} - \varepsilon_2 \vec{E}_{2,n} = \rho_s}$$

In the absence of charge (i.e. $\rho_s = 0$)

$$\boxed{\varepsilon_1 \vec{E}_{1,n} = \varepsilon_2 \vec{E}_{2,n} \qquad (\rho_s = 0)}$$

The normal component of $\vec{E}$ is NOT continuous across the interface, but the product of $\varepsilon \vec{E}_n$ is.

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Boundary Conditions for Dielectric-Conductor Interface

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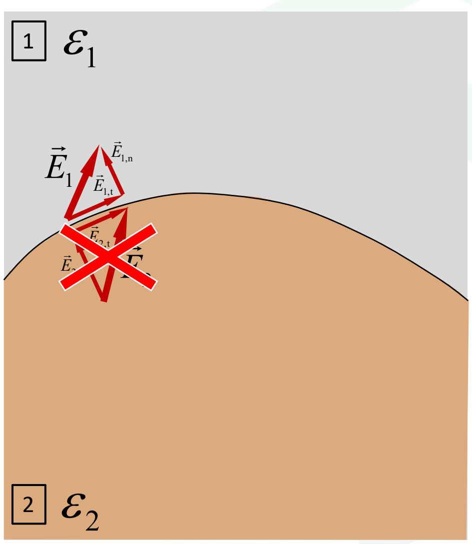
Analysis Setup



Start with same setup as the dielectric-dielectric interface.

Assume the conductor is perfect.

$$\sigma \rightarrow \infty$$

Recall Ohm's law

$$\vec{J} = \sigma \vec{E}$$

In order for $\vec{J}$ not to be infinite, $\vec{E} = 0$ inside the conductor.

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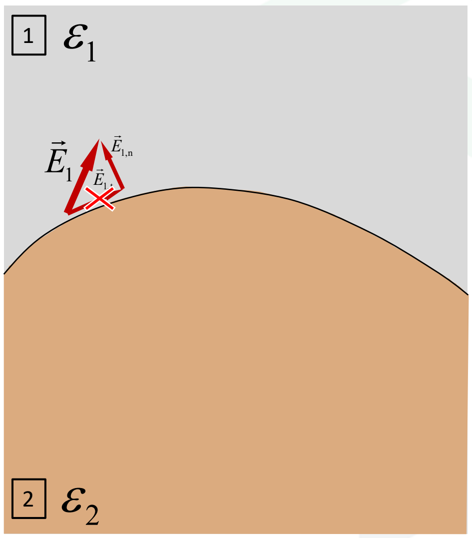
Analysis Setup



If $E_{2,t}$ is zero, then

$$E_{1,t} = 0$$

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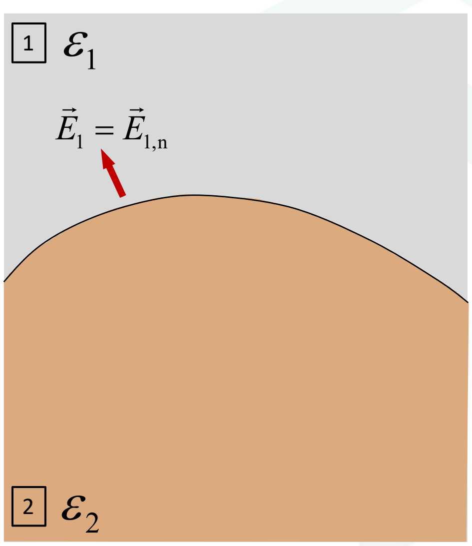
Analysis Setup



There can only be a normal component for the electric field at the interface with a perfect conductor.

$$\vec{E}_1 = E_{1,n}\hat{a}_n$$

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Notes About Perfect Conductors

- No electric field can exist inside of a perfect conductor (i.e. $\vec{E} = 0$).
- Electric potential V is constant throughout a perfect conductor (i.e. $\nabla^2 V = 0$).
- The electric field at the boundary with a perfect conductor has no tangential component. The electric field can only be normal at the interface to a metal.

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