Electromagnetics:
Electromagnetic Field Theory

Laplace's Equation

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Outline

- Derivation of Laplace's Equation
- Meaning of Laplace's Equation

$$\nabla^2 V = 0$$

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Derivation of Laplace's Equation

Derivation of Poisson's Equation (1 of 2)

In electrostatics, the field around charges is described by Gauss' law

$$\nabla \bullet \vec{D} = \rho_v$$

In LI media, the constitutive relation is $\vec{D} = \varepsilon \vec{E}$ so Gauss' law can be written in terms of $\vec{E}$.

$$\nabla \bullet \left(\varepsilon \vec{E} \right) = \rho_v$$

In electrostatics, the electric field is related to electric potential through $\vec{E} = -\nabla V$. This definition can be used to put the above equation solely in terms of the electric potential.

$$\nabla \bullet \left[\varepsilon \left(-\nabla V \right) \right] = \rho_v$$

Derivation of Poisson's Equation (2 of 2)

The previous slides leads to *Poisson's equation for inhomogeneous media*

$$\nabla \bullet \left[\varepsilon\left(-\nabla V\right)\right] = \rho_{\mathrm{v}} \longrightarrow \boxed{\nabla \bullet \left[\varepsilon\left(\nabla V\right)\right] = -\rho_{\mathrm{v}}}$$

Poisson's equation for inhomogeneous media

If the medium is homogeneous, ε is a constant and can be brought to the righthand side of the equation.

$$\nabla \bullet \left(\nabla V\right) = -\frac{\rho_{\mathrm{v}}}{\varepsilon} \longrightarrow \boxed{\nabla^2 V = -\frac{\rho_{\mathrm{v}}}{\varepsilon}}$$

Poisson's equation for homogeneous media

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Derivation of Laplace's Equation

In the absence of charge, $\rho_{\mathrm{v}} = 0$ and Poisson's equation reduces to Laplace's equation.

$$\nabla \bullet \left[\varepsilon\left(\nabla V\right)\right] = -\rho_{\mathrm{v}} \longrightarrow \boxed{\nabla \bullet \left[\varepsilon\left(\nabla V\right)\right] = 0}$$

Laplace's equation for inhomogeneous media

∇^2 is called "the Laplacian"

$$\nabla^2 V = -\frac{\rho_{\mathrm{v}}}{\varepsilon} \longrightarrow \boxed{\nabla^2 V = 0}$$

Laplace's equation for homogeneous media

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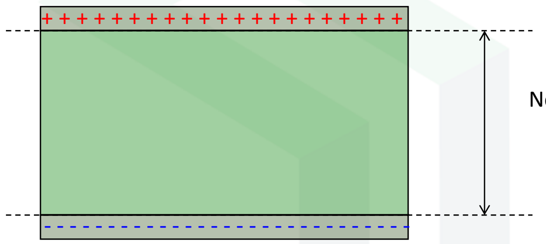
No Charge in Electrostatics?



No charge in the region between the plates.

$$\nabla^2 V = 0$$

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Notes

Equation Name	Inhomogeneous	Homogeneous
Poisson's Equation	$\nabla \bullet \left[\varepsilon\left(\nabla V\right)\right] = -\rho_v$	$\nabla^2 V = -\frac{\rho_v}{\varepsilon}$
Laplace's Equation	$\nabla \bullet \left[\varepsilon\left(\nabla V\right)\right] = 0$	$\nabla^2 V = 0$

- Poisson's and Laplace's equations describe how electric potential varies throughout a volume.
- These are scalar differential equations and usually easier to solve than vector differential equations.
- Use Poisson's equation when there is charge and Laplace's equation when there is not.
- Laplace's equation is particularly important in electrostatics because it can be used to calculate electric potential around conductors maintained at different voltages.
- *Uniqueness theorem* states that there exists only one solution.

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Meaning of Laplace’s Equation

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Meaning of Laplace’s Equation

Laplace’s equation is

$$\nabla^2 u = 0$$

∇^2 is a 3D second-order derivative. ➡ A second-order derivative quantifies curvature. ➡ But, we set the second-order derivative to zero.

Functions satisfying Laplace’s equation vary linearly.

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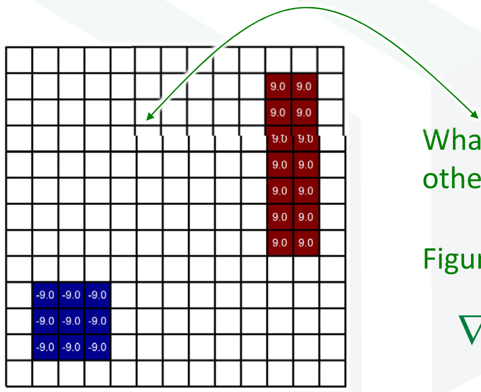
Problem Setup

Suppose the value of $V(x, y)$ is known at some points in space.



What does the function look like at every other point?

Figure it out by solving Laplace's equation.

$$\nabla^2 V(x, y) = 0$$

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Solution of Laplace's Equation

Laplace's equation is sort of a "number filler inner."

0.1	0.4	0.8	1.3	1.8	2.5	3.2	4.1	5.0	6.0	6.9	7.0	6.4
-0.2	0.1	0.5	1.0	1.7	2.4	3.3	4.3	5.5	6.9	9.0	9.0	7.4
-0.6	-0.3	0.1	0.7	1.4	2.2	3.2	4.4	5.7	7.2	9.0	9.0	7.8
-1.1	-0.8	-0.4	0.2	1.0	1.9	3.0	4.3	5.7	7.3	9.0	9.0	7.9
-1.7	-1.5	-1.0	-0.4	0.4	1.4	2.6	4.0	5.5	7.2	9.0	9.0	7.9
-2.4	-2.3	-1.9	-1.3	-0.4	0.8	2.1	3.5	5.2	7.0	9.0	9.0	7.8
-3.3	-3.3	-3.0	-2.4	-1.3	-0.1	1.3	2.9	4.7	6.7	9.0	9.0	7.6
-4.4	-4.7	-4.5	-3.8	-2.6	-1.1	0.5	2.1	3.9	6.0	9.0	9.0	7.1
-5.6	-6.4	-6.4	-5.8	-4.1	-2.2	-0.5	1.2	2.9	4.5	6.0	6.4	5.8
-6.8	-9.0	-9.0	-9.0	-5.6	-3.3	-1.3	0.3	1.8	3.1	4.2	4.6	4.6
-7.1	-9.0	-9.0	-9.0	-6.1	-3.8	-2.0	-0.4	0.9	2.0	2.9	3.4	3.5
-7.0	-9.0	-9.0	-9.0	-6.1	-4.0	-2.3	-0.9	0.2	1.2	2.0	2.4	2.7
-6.1	-6.9	-7.0	-6.6	-5.2	-3.7	-2.4	-1.2	-0.2	0.6	1.3	1.8	2.1

Laplace's equation fills in the numbers so they vary linearly between known regions.

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Another Example

X	X	X	X	X	X	X	X	X	X	X	X	X
X	X	1.0	1.7	2.3	2.9	3.5	4.0	4.6	5.1	X	X	X
X	X	1.5							5.5	X	X	X
X	X	2.0							6.0	X	X	X
X	X	2.5							6.5	X	X	X
X	X	3.1							7.0	X	X	X
X	X	3.6							7.5	X	X	X
X	X	4.2							8.0	X	X	X
X	X	4.8							8.5	X	X	X
X	X	5.4	6.0	6.6	7.1	7.7	8.1	8.6	9.0	X	X	X
X	X	X	X	X	X	X	X	X	X	X	X	X
X	X	X	X	X	X	X	X	X	X	X	X	X
X	X	X	X	X	X	X	X	X	X	X	X	X

X	X	X	X	X	X	X	X	X	X	X	X	X
X	X	1.0	1.7	2.3	2.9	3.5	4.0	4.6	5.1	X	X	X
X	X	1.5	2.1	2.8	3.4	3.9	4.5	5.0	5.5	X	X	X
X	X	2.0	2.6	3.2	3.8	4.4	5.0	5.5	6.0	X	X	X
X	X	2.5	3.1	3.8	4.3	4.9	5.4	6.0	6.5	X	X	X
X	X	3.1	3.7	4.3	4.9	5.4	6.0	6.5	7.0	X	X	X
X	X	3.6	4.2	4.8	5.4	5.9	6.5	7.0	7.5	X	X	X
X	X	4.2	4.8	5.4	6.0	6.5	7.0	7.5	8.0	X	X	X
X	X	4.8	5.4	6.0	6.5	7.1	7.6	8.0	8.5	X	X	X
X	X	5.4	6.0	6.6	7.1	7.7	8.1	8.6	9.0	X	X	X
X	X	X	X	X	X	X	X	X	X	X	X	X
X	X	X	X	X	X	X	X	X	X	X	X	X
X	X	X	X	X	X	X	X	X	X	X	X	X

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