Electromagnetics:
Electromagnetic Field Theory

Solving Laplace's Equation

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Recipe for Solving Laplace's Equation

Laplace's equation is solved as a boundary value problem (i.e. partial differential equation plus boundary conditions).

1. Choose a coordinate system that will simplify the math.
2. Solve Laplace's equation $\nabla^2 V = 0$ in each homogeneous region.
   a. When V is a function of only one variable, use direct integration.
   b. Otherwise, use separation of variables.
3. Apply the boundary conditions at the edges of the homogeneous regions.
4. Calculate $\vec{E}$ from V using $\vec{E} = -\nabla V$.
5. Calculate $\vec{D}$ from $\vec{E}$ using $\vec{D} = \varepsilon\vec{E}$.

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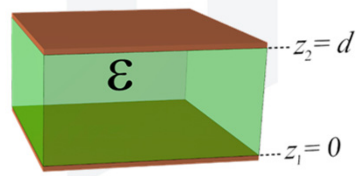
Example #1 – Voltage Between the Plates of a Capacitor

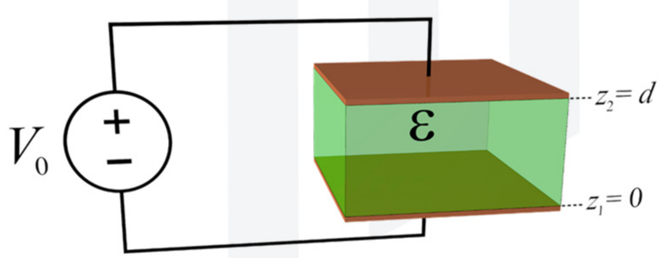
Suppose there exists a medium with permittivity ε and thickness d.



Example #1 – Voltage Between the Plates of a Capacitor

Suppose there exists a medium with permittivity ε and thickness d.

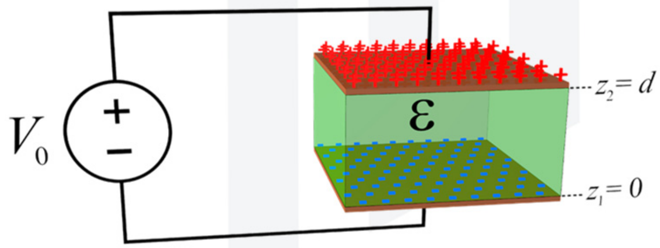
Then apply a voltage V_0 across that medium.



Example #1 – Voltage Between the Plates of a Capacitor

Suppose there exists a medium with permittivity ε and thickness d.

Then apply a voltage V_0 across that medium, which puts charge on the plates.



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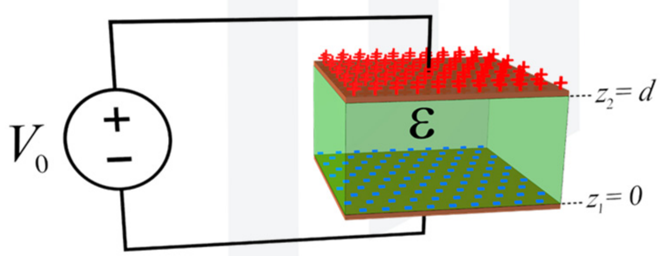
Example #1 – Voltage Between the Plates of a Capacitor

Suppose there exists a medium with permittivity ε and thickness d.

Then apply a voltage V_0 across that medium, which puts charge on the plates.

Calculate the electric potential and electric field between the plates.



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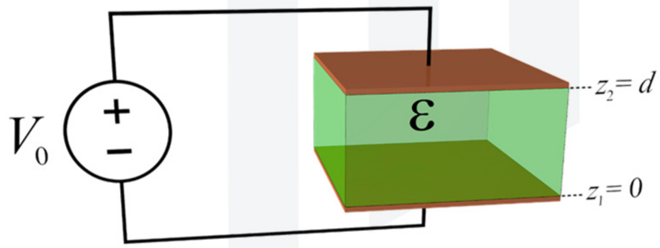
Example #1 – Voltage Between the Plates of a Capacitor

Step 1 – Choose a coordinate system.

Cartesian



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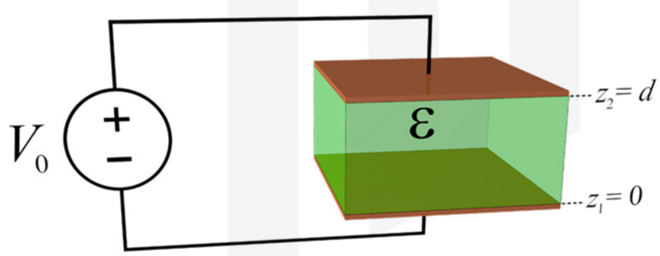
Example #1 – Voltage Between the Plates of a Capacitor

Step 2 – Solve Laplace's equation

$$\nabla^2 V = 0$$
$$V(0) = 0$$
$$V(d) = V_0$$

If we assume the device is uniform in the x and y directions, Laplace's equation reduces to

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad \rightarrow \quad \frac{d^2 V}{dz^2} = 0$$



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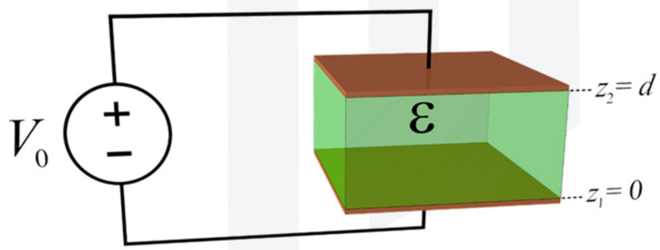
Example #1 – Voltage Between the Plates of a Capacitor

Step 2 – Solve Laplace's equation

Integrate to get

$$\frac{d^2V}{dz^2}=0 \quad \rightarrow \quad V(z)=az+b$$



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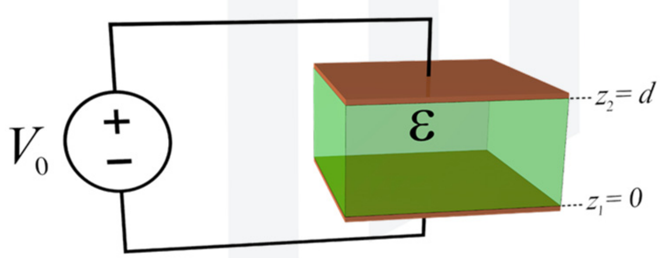
Example #1 – Voltage Between the Plates of a Capacitor

Step 3 – Apply boundary conditions.

First boundary condition…

$$V(0)=0 \quad \rightarrow \quad V(0)=a\cdot 0+b=\underline{b=0}$$



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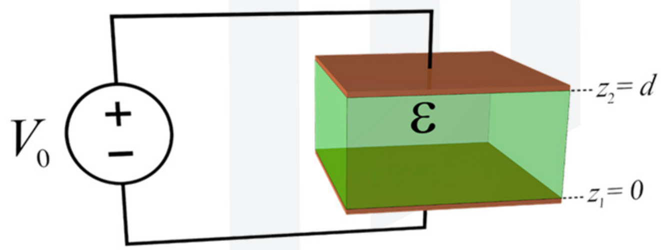
Example #1 – Voltage Between the Plates of a Capacitor

Step 3 – Apply boundary conditions.

Second boundary condition…

$$V(d)=V_0 \quad \rightarrow \quad V(d)=a\cdot d+0=V_0 \quad \rightarrow \quad a=\frac{V_0}{d}$$

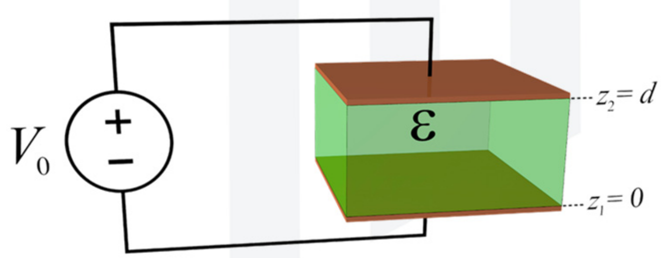


Example #1 – Voltage Between the Plates of a Capacitor

Step 3 – Apply boundary conditions.

Altogether, the solution is

$$V(z)=\frac{V_0}{d}z \qquad 0\le z\le d$$



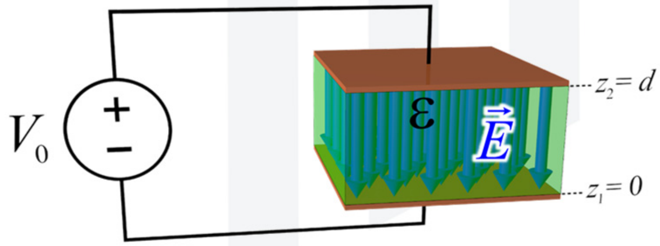
Example #1 – Voltage Between the Plates of a Capacitor

Step 4 – Calculate $\vec{E}$ from V.

The electric field intensity is

$$\vec{E} = -\nabla V \quad \rightarrow \quad E_z = -\frac{d}{dz}V \quad \rightarrow \quad E_z = -\frac{d}{dz}\left(\frac{V_0}{d}z\right) = -\frac{V_0}{d} \quad \rightarrow \quad \boxed{\vec{E} = -\frac{V_0}{d}\hat{a}_z}$$

Observe that $\vec{E}$ does not depend on ε.



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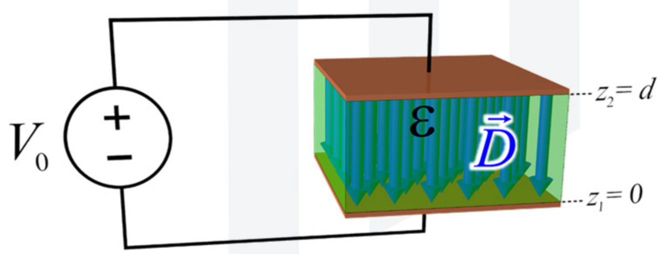
Example #1 – Voltage Between the Plates of a Capacitor

Step 5 – Calculate $\vec{D}$ from $\vec{E}$.

Apply the constitutive relation to get the electric flux density

$$\vec{D} = \varepsilon\vec{E} \quad \rightarrow \quad \vec{D} = -\varepsilon\frac{V_0}{d}\hat{a}_z \quad \rightarrow \quad \boxed{\vec{D} = -\frac{\varepsilon V_0}{d}\hat{a}_z}$$



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