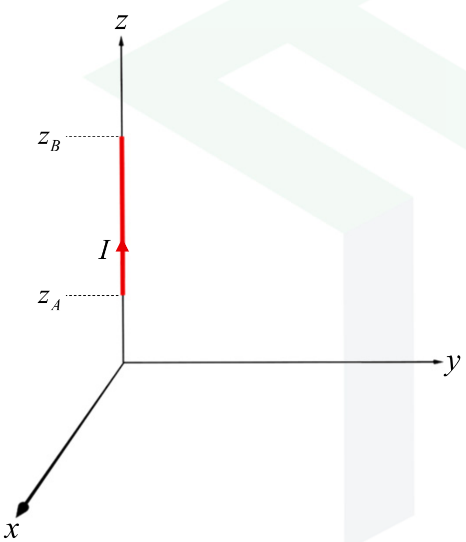
Electromagnetics:

Electromagnetic Field Theory

Example 1 – Magnetic Field Around a Finite-Length Wire

1

Example #1 – Finite Length Wire



What is the magnetic field $\vec{H}$?

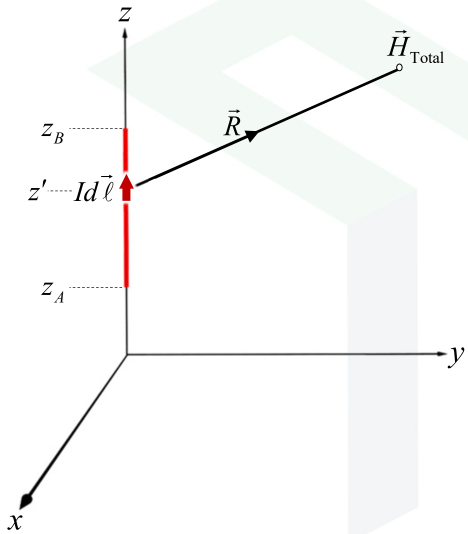
The total magnetic field is obtained by integrating the Biot-Savart law.

$$\vec{H} = \int_L d\vec{H} = \int_{z_A}^{z_B} \frac{I d\vec{\ell} \times \hat{a}_R}{4\pi R^2} = \int_{z_A}^{z_B} \frac{I d\vec{\ell} \times \vec{R}}{4\pi R^3}$$

EMPossible Slide 2

2

Example #1 – Finite Length Wire



What is the magnetic field $\vec{H}$?

For this problem

$$d\vec{\ell} = dz\hat{a}_z \qquad \vec{R} = \rho\hat{a}_\rho + (z - z')\hat{a}_z$$

The cross product becomes

$$\begin{aligned} d\vec{\ell}\times\vec{R} &= (dz\hat{a}_z)\times\left[\rho\hat{a}_\rho + (z-z')\hat{a}_z\right] \\ &= \rho dz\hat{a}_\phi \end{aligned}$$

Putting this back into the integral gives

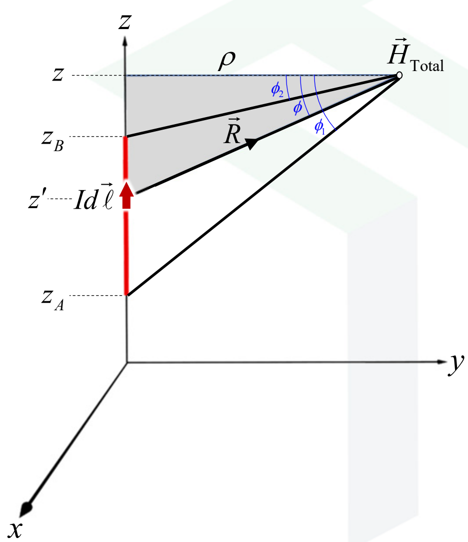
$$\vec{H} = \int_{z_A}^{z_B} \frac{I\rho dz\hat{a}_\phi}{4\pi R^3} = \frac{\rho I}{4\pi}\hat{a}_\phi \int_{z_A}^{z_B} \frac{dz}{R^3}$$

EMPossible

Slide 3

3

Example #1 – Finite Length Wire



What is the magnetic field $\vec{H}$?

Instead of integrating over z, integrate over angle ϕ.

$$z_A \rightarrow \phi_1$$
$$z_B \rightarrow \phi_2$$
$$dz' \rightarrow ?$$
$$\vec{R}\big/\left|\vec{R}\right|^3 \rightarrow ?$$

From the figure, it can be seen that

$$\tan\phi = \frac{z - z'}{\rho} \quad \rightarrow \quad z' = z - \rho\tan\phi$$

Differentiate this expression to get

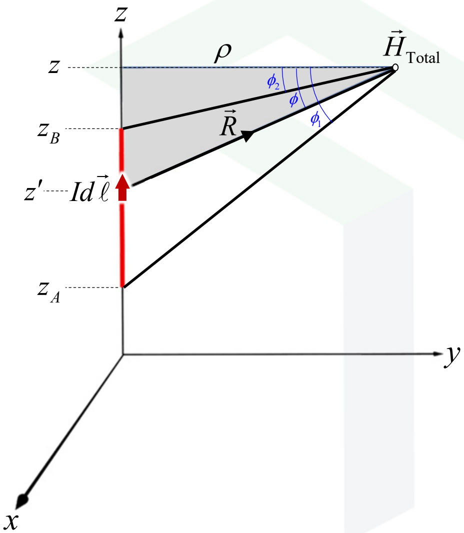
$$dz' = -\rho\sec^2\phi\, d\phi$$

EMPossible

Slide 4

4

Example #1 – Finite Length Wire



What is the magnetic field $\vec{H}$?

The vector $\vec{R}$ is

$$\begin{aligned}\vec{R} &= \rho\hat{a}_\rho + (z - z')\hat{a}_z \\ &= \rho\hat{a}_\rho + \rho\tan\phi\,\hat{a}_z \\ &= \frac{\rho}{\cos\phi}\left(\cos\phi\,\hat{a}_\rho + \sin\phi\,\hat{a}_z\right) \\ &= \rho\sec\phi\underbrace{\left(\cos\phi\,\hat{a}_\rho + \sin\phi\,\hat{a}_z\right)}_{\text{Magnitude is 1}}\end{aligned}$$

An expression can now be written for R^3.

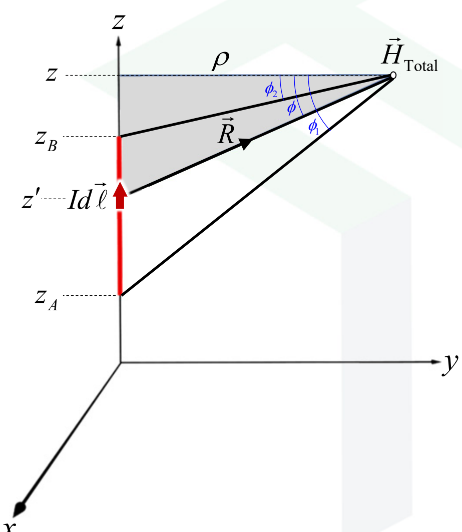
$$\begin{aligned}R^3 &= \left|\vec{R}\right|^3 \\ &= \left|\rho\sec\phi\left(\cos\phi\,\hat{a}_\rho + \sin\phi\,\hat{a}_z\right)\right|^3 \\ &= \left(\rho\sec\phi\right)^3\end{aligned}$$

EMPossible

Slide 5

5

Example #1 – Finite Length Wire



What is the magnetic field $\vec{H}$?

The integral in terms of ϕ becomes

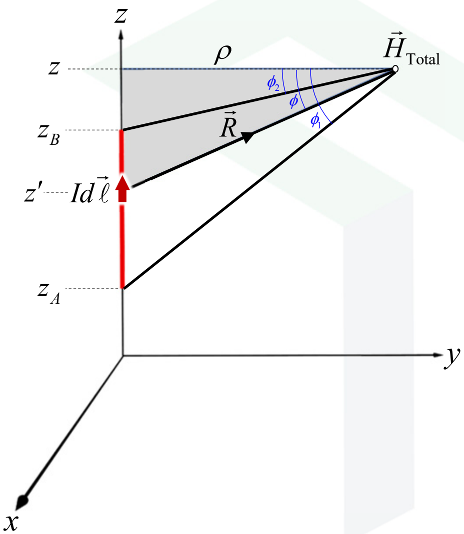
$$\begin{aligned}\vec{H} &= \frac{\rho I}{4\pi}\hat{a}_\phi\int_{z_A}^{z_B}\frac{dz}{\left(\rho\sec\phi\right)^3} \\ &= \frac{\rho I}{4\pi}\hat{a}_\phi\int_{\phi_1}^{\phi_2}\frac{\left(-\rho\sec^2\phi\,d\phi\right)}{\left(\rho\sec\phi\right)^3} \\ &= -\frac{I}{4\pi\rho}\hat{a}_\phi\int_{\phi_1}^{\phi_2}\frac{d\phi}{\sec\phi} \\ &= -\frac{I}{4\pi\rho}\hat{a}_\phi\int_{\phi_1}^{\phi_2}\cos\phi\,d\phi\end{aligned}$$

EMPossible

Slide 6

6

Example #1 – Finite Length Wire



What is the magnetic field $\vec{H}$?

The integral can now be evaluated as

$$\vec{H} = -\frac{I}{4\pi\rho}\hat{a}_\phi \int_{\phi_1}^{\phi_2} \cos\phi \, d\phi$$

$$= -\frac{I}{4\pi\rho}\hat{a}_\phi \left(\sin\phi \Big|_{\phi_1}^{\phi_2}\right)$$

$$= -\frac{I}{4\pi\rho}\hat{a}_\phi \left(\sin\phi_2 - \sin\phi_1\right)$$

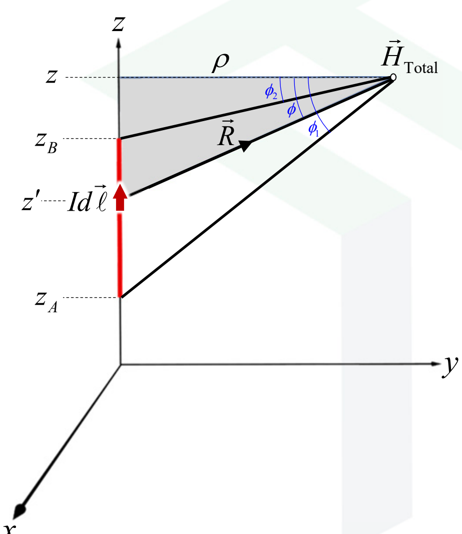
$$= \frac{I}{4\pi\rho}\left(\sin\phi_1 - \sin\phi_2\right)\hat{a}_\phi$$

$$\boxed{\vec{H} = \frac{I}{4\pi\rho}\left(\sin\phi_1 - \sin\phi_2\right)\hat{a}_\phi}$$

EMPossible Slide 7

7

Example #1 – Finite Length Wire



Observations about the solution:

$$\vec{H} = \frac{I}{4\pi\rho}\left(\sin\phi_1 - \sin\phi_2\right)\hat{a}_\phi$$

1. This solution is applicable to any straight wire with uniform current.
2. Magnitude of $\vec{H}$ decays as $1/\rho$.
3. Magnetic has only an $\hat{a}_\phi$ component. This means the magnetic field forms loops around the wire.

EMPossible Slide 8

8