Electromagnetics:
Electromagnetic Field Theory

Example – The Solenoid Inductor

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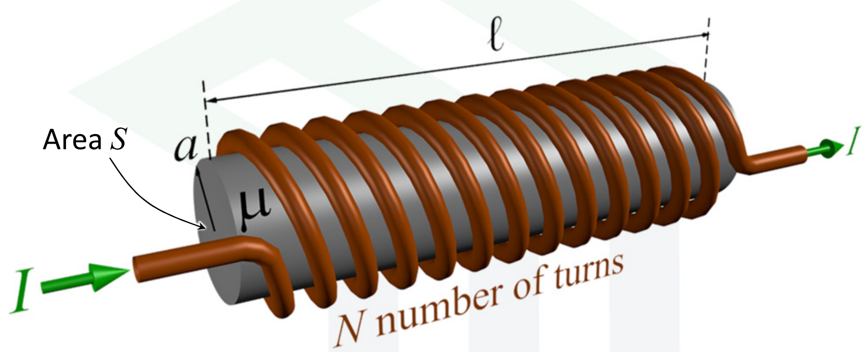
The Solenoidal Inductor



$$L = \frac{\mu N^2 S}{\ell} \quad \text{Henries}$$

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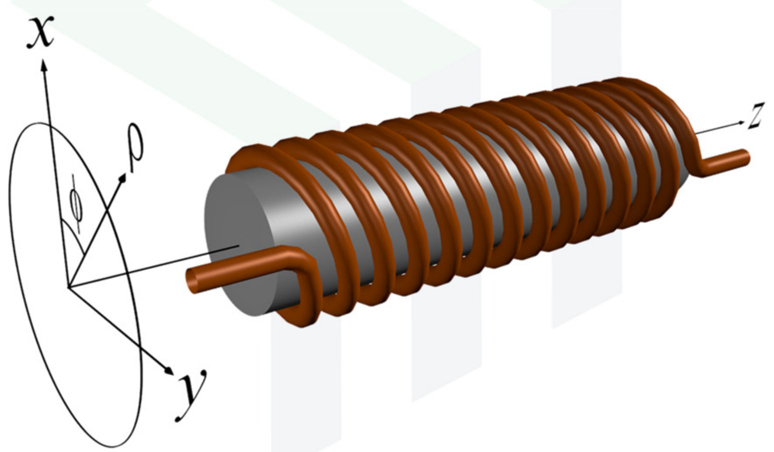
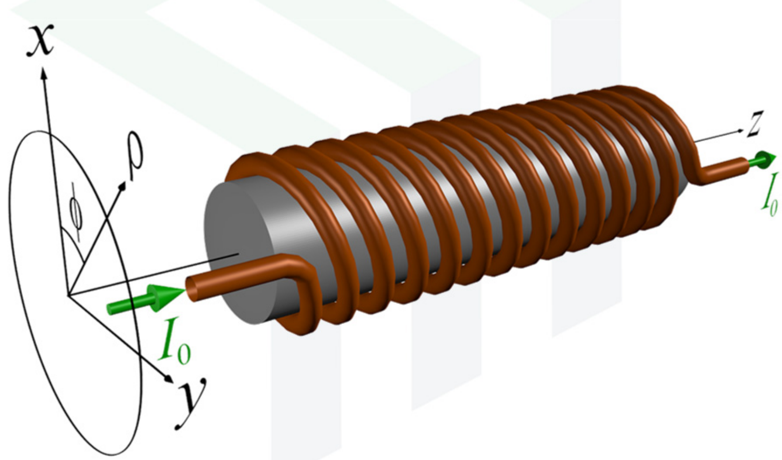
Step 1 – Choose a Coordinate System

Cylindrical coordinates (ρ, ϕ, z) seem appropriate.



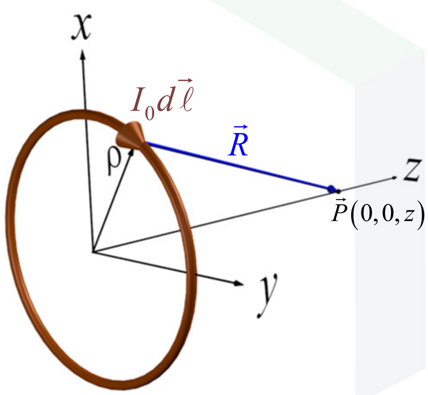
Step 2 – Let Inductor Carry Current I_0

That was easy!



Step 3 – Calculate $\vec{H}$

Calculate the magnetic field $\vec{H}$ due to a single loop and then extend that answer to N loops.



Write the Biot-Savart law at point (0,0,z).

$$d\vec{H} = \frac{I_0 d\vec{\ell} \times \vec{R}}{4\pi \left|\vec{R}\right|^3}$$

The terms in this equation are

$$d\vec{\ell} = a d\phi \hat{a}_\phi$$

$$\vec{R} = -a\hat{a}_\rho + z\hat{a}_z$$

$$\begin{aligned} d\vec{\ell} \times \vec{R} &= \left(a d\phi \hat{a}_\phi\right) \times \left(-a\hat{a}_\rho + z\hat{a}_z\right) \\ &= -a^2 d\phi\left(\hat{a}_\phi \times \hat{a}_\rho\right) + azd\phi\left(\hat{a}_\phi \times \hat{a}_z\right) \\ &= a^2 d\phi \hat{a}_z + azd\phi \hat{a}_\rho \end{aligned}$$

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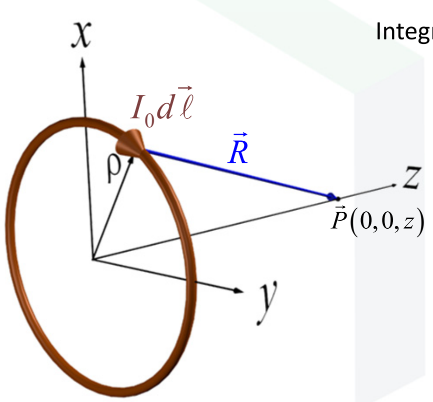
Step 3 – Calculate $\vec{H}$

Last, $\left|\vec{R}\right|^3$ is needed.

$$\vec{R} = -a\hat{a}_\rho + z\hat{a}_z$$

$$\left|\vec{R}\right|^3 = \left(a^2 + z^2\right)^{3/2}$$



Now a big ugly expression can be written for $d\vec{H}_1$.

$$d\vec{H}_1 = \frac{I_0 d\vec{\ell} \times \vec{R}}{4\pi \left|\vec{R}\right|^3} = \frac{I_0\left(a^2 d\phi \hat{a}_z + azd\phi \hat{a}_\rho\right)}{4\pi\left(a^2 + z^2\right)^{3/2}}$$

Integrate this around the loop to find $\vec{H}_1$.

$$\begin{aligned} \vec{H}_1(z) &= \int_0^{2\pi} d\vec{H}_1 \\ &= \int_0^{2\pi} \frac{I_0\left(a^2 d\phi \hat{a}_z + azd\phi \hat{a}_\rho\right)}{4\pi\left(a^2 + z^2\right)^{3/2}} \\ &= \int_0^{2\pi} \frac{I_0 a^2 d\phi \hat{a}_z}{4\pi\left(a^2 + z^2\right)^{3/2}} + \int_0^{2\pi} \frac{I_0 azd\phi \hat{a}_\rho}{4\pi\left(a^2 + z^2\right)^{3/2}} \end{aligned}$$

Due to symmetry, the second integral equals zero.

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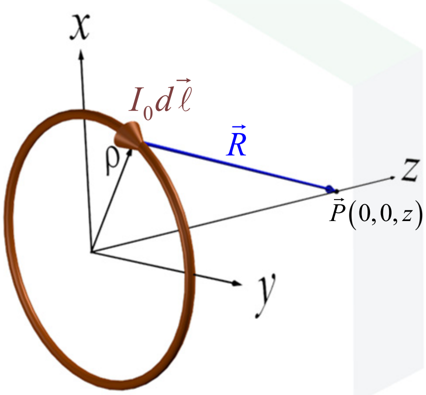
Step 3 – Calculate $\vec{H}$

Finally, the magnetic field along the z-axis is

$$\vec{H}_1 = \int_0^{2\pi} \frac{I_0 a^2 d\phi \hat{a}_z}{4\pi\left(a^2+z^2\right)^{3/2}} = \frac{I_0 a^2 \hat{a}_z}{4\pi\left(a^2+z^2\right)^{3/2}} \underbrace{\int_0^{2\pi} d\phi}_{=2\pi}$$

$$= \frac{I_0 a^2 \hat{a}_z}{4\pi\left(a^2+z^2\right)^{3/2}} 2\pi$$

$$= \frac{I_0 a^2}{2\left(a^2+z^2\right)^{3/2}} \hat{a}_z$$



Remember, this is just the magnetic field for a single loop.

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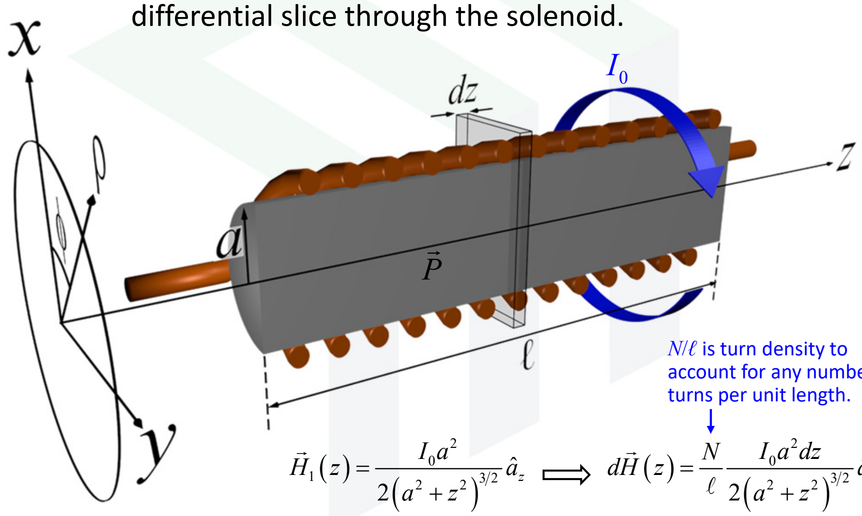
Step 3 – Calculate $\vec{H}$

Think of the expression we just derived as a differential $\vec{H}_1$ due to a differential slice through the solenoid.



N/ℓ is turn density to account for any number of turns per unit length.

$$\vec{H}_1(z) = \frac{I_0 a^2}{2\left(a^2+z^2\right)^{3/2}} \hat{a}_z \implies d\vec{H}(z) = \frac{N}{\ell} \frac{I_0 a^2 dz}{2\left(a^2+z^2\right)^{3/2}} \hat{a}_z$$

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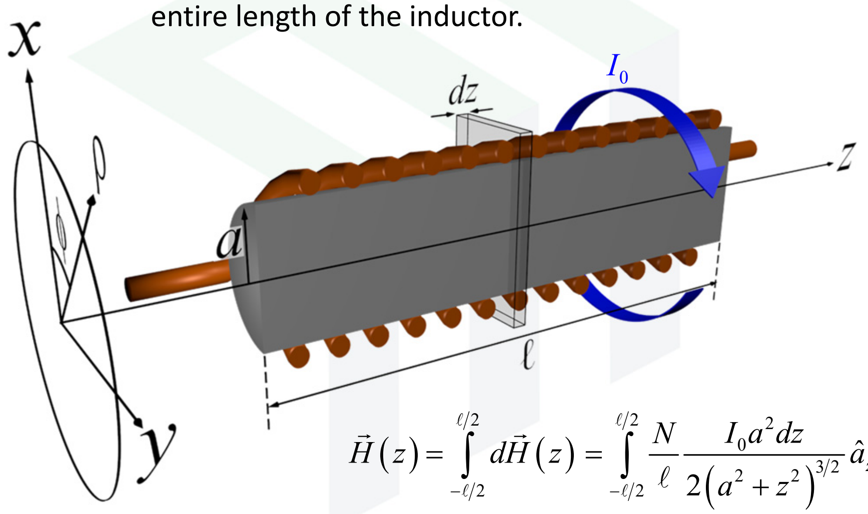
Step 3 – Calculate $\vec{H}$

The total magnetic field $\vec{H}$ is found by integrating $d\vec{H}_z$ through the entire length of the inductor.



$$\vec{H}(z) = \int_{-\ell/2}^{\ell/2} d\vec{H}(z) = \int_{-\ell/2}^{\ell/2} \frac{N}{\ell} \frac{I_0 a^2 dz}{2\left(a^2 + z^2\right)^{3/2}} \hat{a}_z$$

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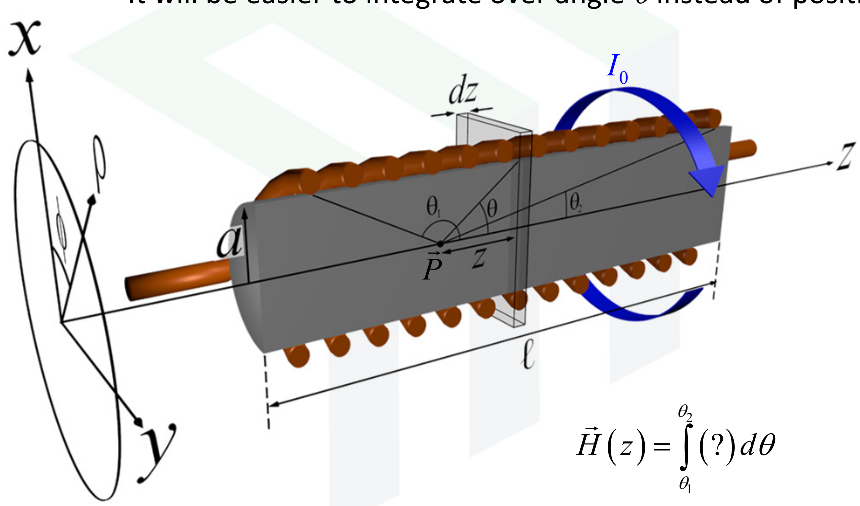
Step 3 – Calculate $\vec{H}$

It will be easier to integrate over angle θ instead of position z.



$$\vec{H}(z) = \int_{\theta_1}^{\theta_2} (?)\, d\theta$$

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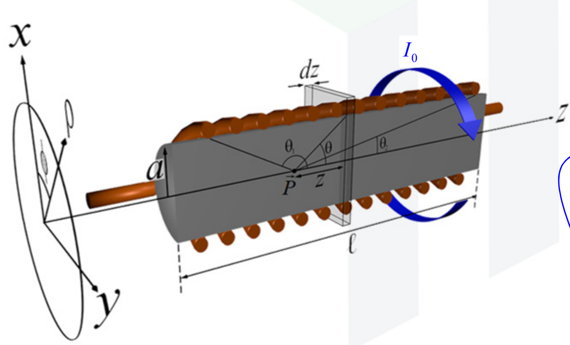
Step 3 – Calculate $\vec{H}$

Angle θ must be related to z. Observe that

$$\tan\theta = \frac{a}{z} \quad \rightarrow \quad z = \frac{a}{\tan\theta} \quad \rightarrow \quad dz = -\frac{a}{\sin^2\theta}d\theta = -\frac{a}{\sin^3\theta}\sin\theta d\theta$$



Now observe that

$$\sin\theta = \frac{a}{\sqrt{a^2+z^2}}$$

$$\sin^3\theta = \frac{a^3}{\left(a^2+z^2\right)^{3/2}}$$

This was done so that the $\left(a^2+z^2\right)^{3/2}$ term be canceled out of the integral.

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Step 3 – Calculate $\vec{H}$

The integral in terms of θ is…

$$\vec{H}(z) = \int_{-\ell/2}^{\ell/2} \frac{N}{\ell} \frac{I_0 a^2 dz}{2\left(a^2+z^2\right)^{3/2}} \hat{a}_z$$

$-\ell/2 \rightarrow \theta_1$, $\ell/2 \rightarrow \theta_2$, $dz \rightarrow -\frac{a}{\sin^3\theta}\sin\theta d\theta$, $\left(a^2+z^2\right)^{3/2} \rightarrow \frac{a^3}{\sin^3\theta}$

$$\vec{H}(z) = \int_{\theta_1}^{\theta_2} \frac{N}{\ell} \frac{I_0 a^2 \left(-\frac{a}{\sin^3\theta}\sin\theta d\theta\right)}{2\left(\frac{a^3}{\sin^3\theta}\right)} \hat{a}_z = -\frac{N I_0 \hat{a}_z}{2\ell}\int_{\theta_1}^{\theta_2} \sin\theta d\theta$$

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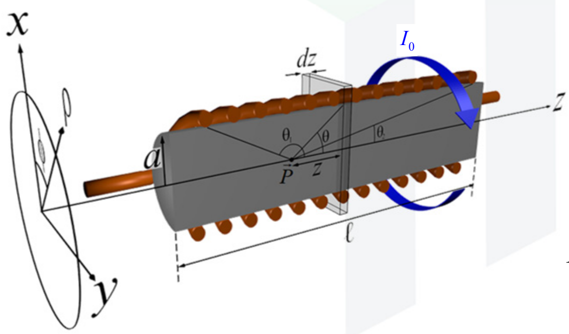
Step 3 – Calculate $\vec{H}$

Finish the integration to get

$$\vec{H}(z) = -\frac{NI_0\hat{a}_z}{2\ell}\int_{\theta_1}^{\theta_2}\sin\theta d\theta = \frac{NI_0\hat{a}_z}{2\ell}\left(\cos\theta_2 - \cos\theta_1\right)$$



For an infinitely long solenoid

$$\theta_1 = 180° \qquad \theta_2 = 0°$$

The solution becomes

$$\vec{H}(z) = \frac{NI_0\hat{a}_z}{2\ell}\left(\cos 0° - \cos 180°\right)$$

$$\vec{H}(z) = \frac{NI_0}{\ell}\hat{a}_z$$

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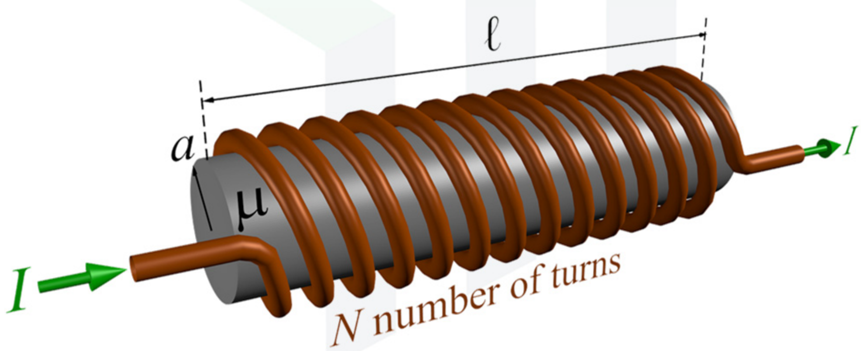
Step 4 – Calculate $\vec{B}$

Given the magnetic field intensity $\vec{H}$, the magnetic flux density $\vec{B}$ is found using the constitutive relation.

$$\vec{B} = \mu\vec{H} = \mu\frac{NI_0}{\ell}\hat{a}_z$$



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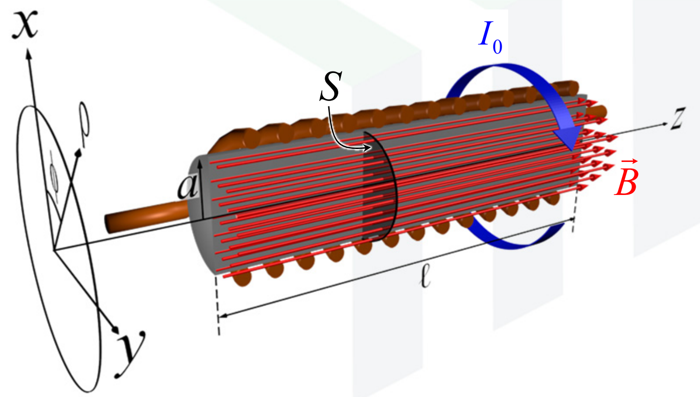
Step 5 – Calculate ψ

The magnetic flux ψ is calculated by integrating $\vec{B}$ in the cross section of the solenoid.

$$\psi = \iint_S \vec{B} \bullet d\vec{s} = B_z S = \frac{\mu N I_0}{\ell} \cdot S$$



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Step 6 – Calculate Inductance L

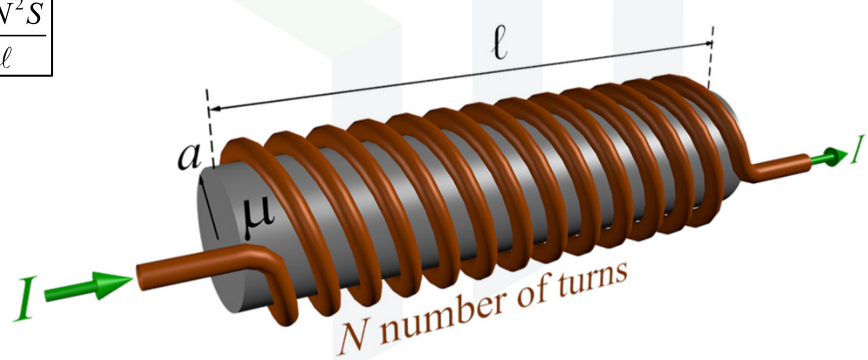
Finally, the inductance L is

$$L = \frac{N\psi}{I_0} = \frac{N \dfrac{\mu N I_0 S}{\ell}}{I_0} = \frac{\mu N^2 S}{\ell}$$

$$\boxed{L = \frac{\mu N^2 S}{\ell}}$$

This is often written as inductance per unit length.

$$\boxed{\frac{L}{\ell} = \mu S \left(\frac{N}{\ell}\right)^2}$$



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