Electromagnetics:
Electromagnetic Field Theory

Example 2 – Deriving Particle Position

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Example #2 – Particle Position

A particle with charge 2.0 mC and mass 8 mg is moving at a velocity of 10 m/s in the positive x direction in the presence of a static magnetic field of $\vec{B} = 4\hat{a}_z \ \mathrm{Wb/m^2}$. If the particle is at the origin at time t = 0, what is the particle's position at t = 3 s?

Solution

Write the Lorentz force equation and include $d\vec{u}/dt$.

$$\vec{F} = m\frac{d\vec{u}}{dt} = Q\left(\vec{E} + \vec{u} \times \vec{B}\right)$$

Solve this for $d\vec{u}/dt$.

$$\frac{d\vec{u}}{dt} = \frac{Q}{m}\left(\vec{E} + \vec{u} \times \vec{B}\right)$$

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Example #2 – Particle Position

Expand the vector quantities.

$$\frac{d\vec{u}}{dt} = \frac{Q}{m}\left(\vec{E} + \vec{u} \times \vec{B}\right)$$

$$\frac{d}{dt}\left(u_x\hat{a}_x + u_y\hat{a}_y + u_z\hat{a}_z\right) = \frac{Q}{m}\left[\left(E_x\hat{a}_x + E_y\hat{a}_y + E_z\hat{a}_z\right) + \left(u_yB_z - u_zB_y\right)\hat{a}_x + \left(u_zB_x - u_xB_z\right)\hat{a}_y + \left(u_xB_y - u_xB_y\right)\hat{a}_z\right]$$

$$\frac{du_x}{dt}\hat{a}_x + \frac{du_y}{dt}\hat{a}_y + \frac{du_z}{dt}\hat{a}_z = \frac{Q}{m}\left(E_x + u_yB_z - u_zB_y\right)\hat{a}_x + \frac{Q}{m}\left(E_y + u_zB_x - u_xB_z\right)\hat{a}_y + \frac{Q}{m}\left(E_z + u_xB_y - u_xB_y\right)\hat{a}_z$$

Extract three scalar equations from this one vector equation.

Example #2 – Particle Position

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Example #2 – Particle Position

Expand the vector quantities.

$$\frac{d\vec{u}}{dt}=\frac{Q}{m}\left(\vec{E}+\vec{u}\times\vec{B}\right)$$

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$$\frac{du_x}{dt}\hat{a}_x+\frac{du_y}{dt}\hat{a}_y+\frac{du_z}{dt}\hat{a}_z=\frac{Q}{m}\left(E_x+u_yB_z-u_zB_y\right)\hat{a}_x+\frac{Q}{m}\left(E_y+u_zB_x-u_xB_z\right)\hat{a}_y+\frac{Q}{m}\left(E_z+u_xB_y-u_xB_y\right)\hat{a}_z$$

Extract three scalar equations from this one vector equation.

$$\frac{du_x}{dt}=\frac{Q}{m}\left(E_x+u_yB_z-u_zB_y\right)$$

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Extract three scalar equations from this one vector equation.

$$\frac{du_x}{dt}=\frac{Q}{m}\left(E_x+u_yB_z-u_zB_y\right)$$

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$$\frac{du_z}{dt}=\frac{Q}{m}\left(E_z+u_xB_y-u_xB_y\right)$$

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Example #2 – Particle Position

Substitute the known values into these equations to simplify them.

$$\frac{du_x(t)}{dt}=\frac{Q}{m}\left[E_x+u_y(t)B_z-u_z(t)B_y\right]=\frac{\left(2\times10^{-3}\ \text{C}\right)}{\left(8\times10^{-3}\ \text{kg}\right)}\left[0+u_y(t)\left(4\ \text{Wb/m}^2\right)-u_z(t)\cdot0\right]\quad=u_y(t)$$

$$\frac{du_y(t)}{dt}=\frac{Q}{m}\left[E_y+u_z(t)B_x-u_x(t)B_z\right]=\frac{\left(2\times10^{-3}\ \text{C}\right)}{\left(8\times10^{-3}\ \text{kg}\right)}\left[0+u_z(t)\cdot0-u_x(t)\left(4\ \text{Wb/m}^2\right)\right]\quad=-u_x(t)$$

$$\frac{du_z(t)}{dt}=\frac{Q}{m}\left[E_z+u_x(t)B_y-u_y(t)B_x\right]=\frac{\left(2\times10^{-3}\ \text{C}\right)}{\left(8\times10^{-3}\ \text{kg}\right)}\left[0+u_x(t)\cdot0-u_y(t)\cdot0\right]\quad=0$$

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Example #2 – Particle Position

Write the differential equation for the x component of velocity.

$$\frac{du_x(t)}{dt}=u_y(t)$$

Differentiate with respect to time.

$$\frac{d}{dt}\left[\frac{du_x(t)}{dt}\right]=\frac{du_y(t)}{dt}$$

Replace du_y/dt using the differential equation for the y component.

$$\frac{du_y(t)}{dt}=-u_x(t)$$

$$\frac{d}{dt}\left[\frac{du_x(t)}{dt}\right]=-u_x(t)$$

Put differential equation in standard form.

$$\frac{d^2u_x(t)}{dt^2}+u_x(t)=0$$

Solve the differential equation.

$$u_x(t)=A\cos t+B\sin t$$

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Example #2 – Particle Position

Substitute the solution back into the original differential equation for the x component to get the solution for $u_y(t)$.

$$\frac{du_x(t)}{dt} = u_y(t)$$

$$\frac{d}{dt}\left(A\cos t + B\sin t\right) = u_y(t)$$

$$-A\sin t + B\cos t = u_y(t)$$

Write the differential equation for the z component of velocity.

$$\frac{du_z(t)}{dt} = 0$$

This is solved by integrating.

$$u_z(t) = C$$

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Example #2 – Particle Position

Apply the boundary conditions to calculate the constants A, B, and C.

At time $t = 0$, $\vec{u}(0) = (10\ \text{m/s})\hat{a}_x$

$$u_x(t) = A\cos t + B\sin t \qquad u_x(0) = 10\ \text{m/s} = A\cos 0 + B\sin 0 = A$$

$$A = 10\ \text{m/s}$$

$$u_y(t) = -A\sin t + B\cos t \qquad u_y(0) = 0 = -A\sin 0 + B\cos 0 = B$$

$$B = 0$$

$$u_z(t) = C \qquad u_z(0) = 0 = C$$

$$C = 0$$

The equations for the components of velocity are now

$$\left.\begin{aligned} u_x(t) &= 10\cos t \\ u_y(t) &= -10\sin t \\ u_z(t) &= 0 \end{aligned}\right\} \quad \vec{u}(t) = 10\cos(t)\hat{a}_x - 10\sin(t)\hat{a}_y$$

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Example #2 – Particle Position

Position as a function of time is calculated by integrating the expression for velocity.

$$\vec{u}(t) = 10\cos(t)\hat{a}_x - 10\sin(t)\hat{a}_y$$

$$\frac{d\vec{r}(t)}{dt} = \vec{u}(t) = 10\cos(t)\hat{a}_x - 10\sin(t)\hat{a}_y$$

$$\vec{r}(t) = \left[10\sin(t) + A\right]\hat{a}_x + \left[10\cos(t) + B\right]\hat{a}_y$$

The constants A and B are determined by applying the initial condition.

$$\vec{r}(0) = 0 = \left[10\sin(0) + A\right]\hat{a}_x + \left[10\cos(0) + B\right]\hat{a}_y = A\hat{a}_x + (10 + B)\hat{a}_y$$

$$A = 0 \qquad B = -10$$

The final expression for position is

$$\vec{r}(t) = 10\sin(t)\hat{a}_x + 10\left[\cos(t) - 1\right]\hat{a}_y$$

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Example #2 – Particle Position

It is finally possible to evaluate the position at t = 3 s.

$$\vec{r}(t) = 10\sin(t)\hat{a}_x + 10\left[\cos(t) - 1\right]\hat{a}_y$$

$$\vec{r}(3) = 10\sin(3)\hat{a}_x + 10\left[\cos(3) - 1\right]\hat{a}_y$$

$$\boxed{\vec{r}(3) = 1.41\hat{a}_x - 19.90\hat{a}_y}$$

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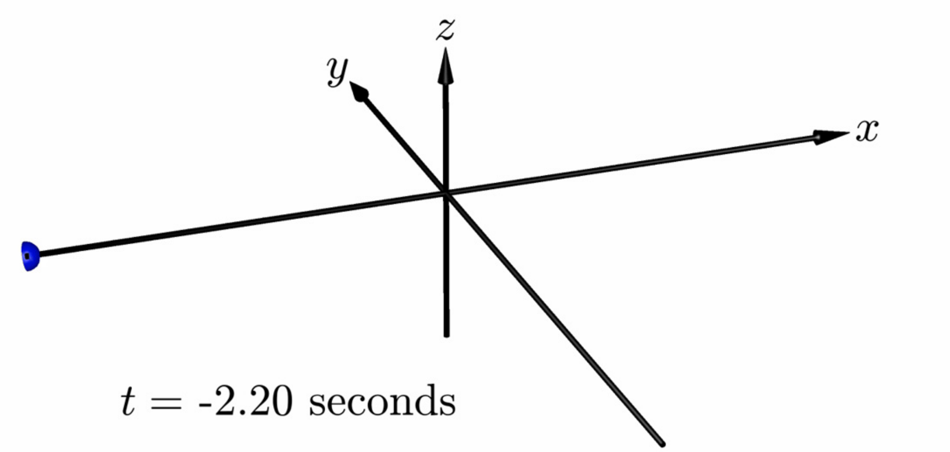
Example #2 – Particle Position



$$\vec{r}(t) = 10\sin(t)\hat{a}_x + 10\left[\cos(t) - 1\right]\hat{a}_y \qquad t \geq 0$$

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