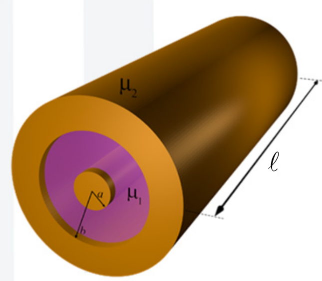
Electromagnetics:
Electromagnetic Field Theory

Example 3 – Inductance of a Coaxial Transmission Line

1

Problem Setup

Derive an expression for the distributed inductance L/ℓ of coaxial line.



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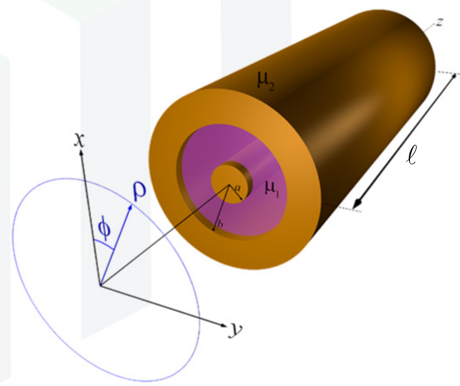
Problem Setup

Derive an expression for the distributed inductance L/ℓ of coaxial line.

Step 1 – Choose a coordinate system.

Cylindrical



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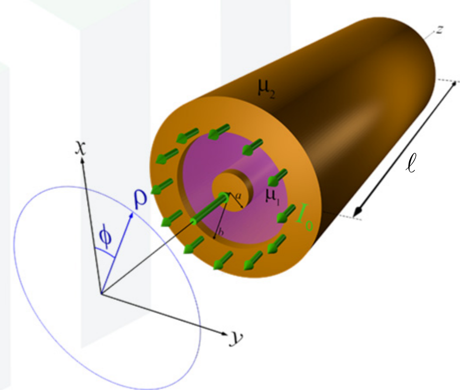
Problem Setup

Step 2 – Let the inductor carry current I_0.

That was easy!



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Problem Setup

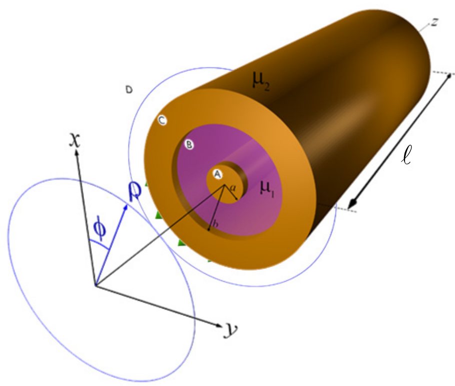
Step 3 – Calculate magnetic field intensity $\vec{H}$.

To do this, analyze the device in each of the four regions A, B, C, and D separately and then stitch together the answers.

A – inner conductor
B – dielectric region
C – outer conductor
D – outside of coax

Calculate the magnetic field using Ampere's circuit law.

$$I = \oint_L \vec{H} \bullet d\vec{\ell} = \iint_S \vec{J} \bullet d\vec{s}$$



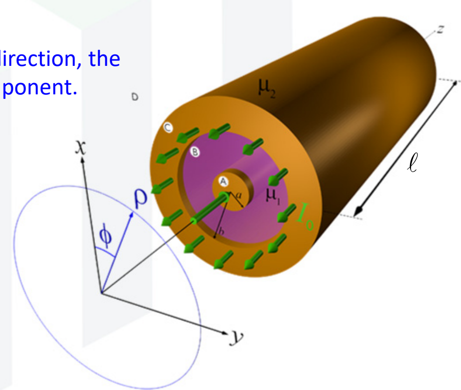
Problem Setup

Step 3 – Calculate magnetic field intensity $\vec{H}$.

Recall the Biot-Savart law which says the magnetic field will be perpendicular to the current and the direction of the observation point.

$$d\vec{H} = \frac{\vec{J}dv \times \hat{a}_R}{4\pi R^2}$$

1. Since the current is solely in the z direction, the magnetic field cannot have a z component.
2. Due to symmetry, the magnetic field will not have a ρ component.
3. The magnetic field will be oriented in the ϕ direction. This is consistent with the magnetic field circulating around currents.



$$\vec{H}(\rho,\phi,z) = H_\phi(\rho,\phi,z)\,\hat{a}_\phi$$

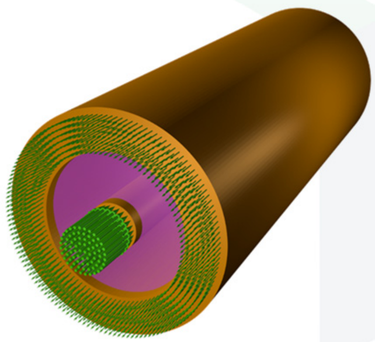
This is what must be found.

Region A – Inner Conductor

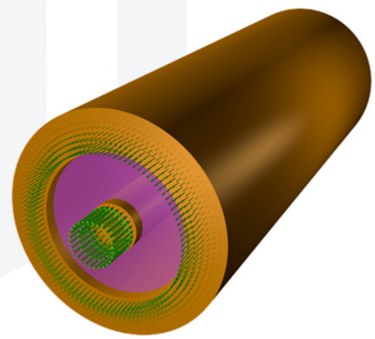
Step 3 – Calculate magnetic field intensity $\vec{H}$.

Assume the current is uniformly distributed throughout the conductors.
This is perfectly valid for magnetostatics, but is a bad approximation at high frequency due to the skin effect.

Current distribution at low frequency



Current distribution at high frequency



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Region A – Inner Conductor

Step 3 – Calculate magnetic field intensity $\vec{H}$.

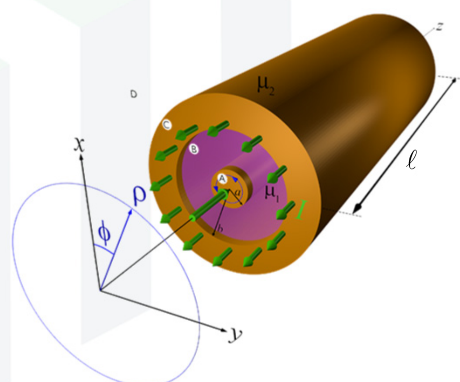
Given this approximation, the current density in the inner conductor is

$$\vec{J}_{\text{inner}} = \frac{I_0}{\pi a^2}\hat{a}_z$$

The total current enclosed within radius ρ is

$$I_A(\rho) = \left(\frac{\text{Area enclosed by } \rho}{\text{Area of inner conductor}}\right) I_0$$

$$= \left(\frac{\pi \rho^2}{\pi a^2}\right) I_0 = I_0 \left(\frac{\rho}{a}\right)^2$$



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Region A – Inner Conductor

Step 3 – Calculate magnetic field intensity $\vec{H}$.

Applying Ampere's circuit law, the current is related to the magnetic field as

$$I_A(\rho) = \oint_L \vec{H} \bullet d\vec{\ell} = \int_0^{2\pi} \left(H_\phi \hat{a}_\phi\right) \bullet \left(\rho d\phi \hat{a}_\phi\right) = \rho H_\phi \int_0^{2\pi} d\phi = 2\pi\rho H_\phi$$

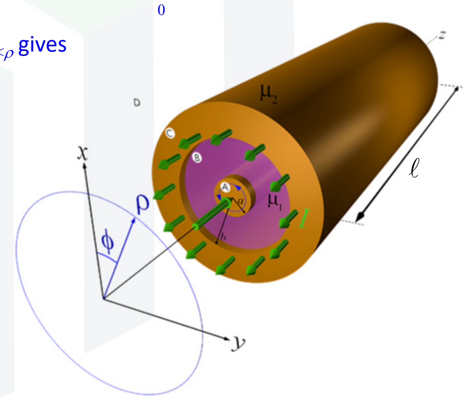
Applying our previous expression for $I_{<\rho}$ gives

$$I_0 \left(\frac{\rho}{a}\right)^2 = 2\pi\rho H_\phi$$

Solve this for H_ϕ to get

$$H_\phi = \frac{I_0 \rho}{2\pi a^2}$$

$$\vec{H}_A(\rho) = \frac{I_0 \rho}{2\pi a^2} \hat{a}_\phi$$



Region B – Dielectric

Step 3 – Calculate magnetic field intensity $\vec{H}$.

Applying Ampere's circuit law, the current is related to the magnetic field as

$$I_B(\rho) = \oint_L \vec{H} \bullet d\vec{\ell} = \int_0^{2\pi} \left(H_\phi \hat{a}_\phi\right) \bullet \left(\rho d\phi \hat{a}_\phi\right) = \rho H_\phi \int_0^{2\pi} d\phi = 2\pi\rho H_\phi$$

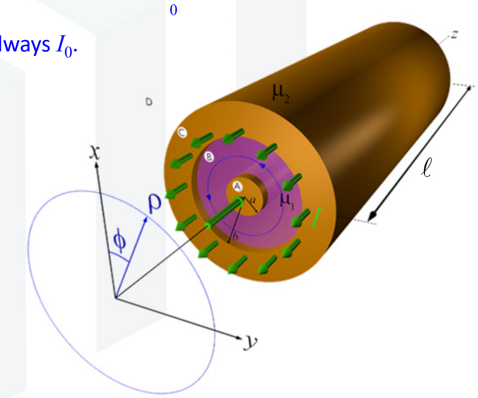
In Region B, the enclosed current is always I_0.

$$I_0 = 2\pi\rho H_\phi$$

Solve this for H_ϕ to get

$$H_\phi = \frac{I_0}{2\pi\rho}$$

$$\vec{H}_B(\rho) = \frac{I_0}{2\pi\rho} \hat{a}_\phi$$



Region C – Outer Conductor

Step 3 – Calculate magnetic field intensity $\vec{H}$.

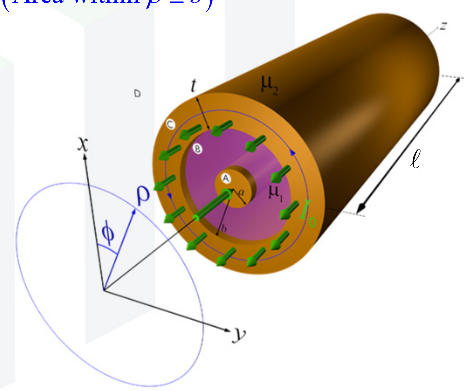
The current density in the outer conductor is

$$\vec{J}_{\text{outer}} = \frac{-I_0}{(\text{Area within } \rho \le b+t) - (\text{Area within } \rho \le b)}\hat{a}_z$$

$$= \frac{-I_0}{\pi(b+t)^2 - \pi b^2}\hat{a}_z$$

$$= -\frac{I_0}{\pi}\frac{1}{b^2 + 2bt + t^2 - b^2}\hat{a}_z$$

$$= -\frac{I_0}{\pi}\frac{1}{t^2 + 2bt}\hat{a}_z$$



Region C – Outer Conductor

Step 3 – Calculate magnetic field intensity $\vec{H}$.

The total current enclosed by radius ρ is

$$I_C(\rho) = I_0 + \iint_C \vec{J}_{\text{outer}} \bullet d\vec{s}$$

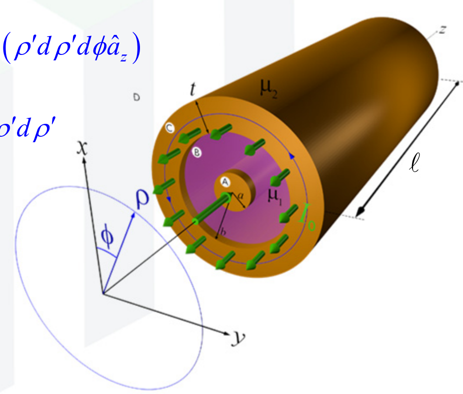
$$= I_0 - \int_0^{2\pi}\int_b^{\rho}\left(\frac{I_0}{\pi}\frac{1}{t^2+2bt}\hat{a}_z\right)\bullet(\rho' d\rho' d\phi \hat{a}_z)$$

$$= I_0 - \frac{I_0}{\pi}\frac{1}{t^2+2bt}\int_b^{\rho}\left(\int_0^{2\pi} d\phi\right)\rho' d\rho'$$

$$= I_0 - \frac{I_0}{\pi}\frac{2\pi}{t^2+2bt}\int_b^{\rho}\rho' d\rho'$$

$$= I_0 - \frac{2I_0}{t^2+2bt}\left.\frac{\rho'^2}{2}\right|_b^{\rho}$$

$$= I_0\left(1 - \frac{\rho^2 - b^2}{t^2+2bt}\right)$$



Region C – Outer Conductor

Step 3 – Calculate magnetic field intensity $\vec{H}$.

Applying Ampere's circuit law, the current is related to the magnetic field as

$$I_C(\rho)=\oint_L \vec{H}\bullet d\vec{\ell}=\int_0^{2\pi}\left(H_\phi\hat{a}_\phi\right)\bullet\left(\rho d\phi\hat{a}_\phi\right)=\rho H_\phi\int_0^{2\pi}d\phi=2\pi\rho H_\phi$$

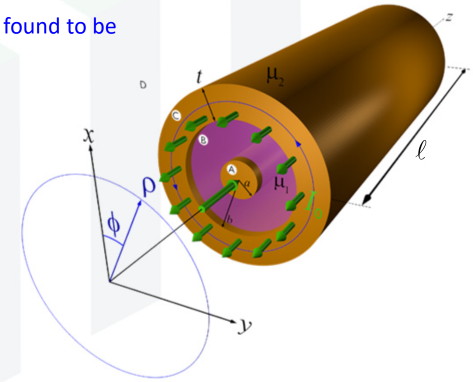
In Region C, the enclosed current was found to be

$$I_C(\rho)=I_0\left(1-\frac{\rho^2-b^2}{t^2+2bt}\right)$$

Putting these expressions together and solving for H_ϕ gives

$$I_0\left(1-\frac{\rho^2-b^2}{t^2+2bt}\right)=2\pi\rho H_\phi$$

$$H_\phi(\rho)=\frac{I_0}{2\pi\rho}\left(1-\frac{\rho^2-b^2}{t^2+2bt}\right)$$



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Region D – Outside Coax

Step 3 – Calculate magnetic field intensity $\vec{H}$.

Applying Ampere's circuit law, the current is related to the magnetic field as

$$I_D(\rho)=\oint_L \vec{H}\bullet d\vec{\ell}=\int_0^{2\pi}\left(H_\phi\hat{a}_\phi\right)\bullet\left(\rho d\phi\hat{a}_\phi\right)=\rho H_\phi\int_0^{2\pi}d\phi=2\pi\rho H_\phi$$

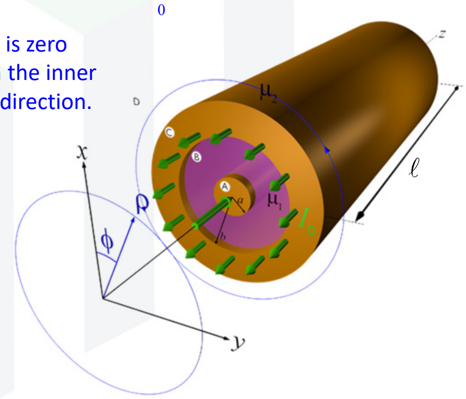
In Region D, the total enclosed current is zero because the same current is present in the inner and outer conductors, but in opposite direction.

$$I_D(\rho)=0$$

Putting these together shows that

$$0=2\pi\rho H_\phi$$

$$H_\phi(\rho)=0$$



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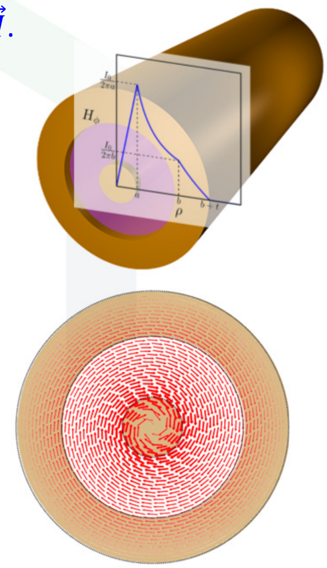
All Together

Step 3 – Calculate magnetic field intensity $\vec{H}$.

All together, the magnetic field intensity is

$$\vec{H}(\rho)=\begin{cases}\dfrac{I_0\rho}{2\pi a^2}\hat{a}_\phi & 0\le\rho\le a\\ \dfrac{I_0}{2\pi\rho}\hat{a}_\phi & a\le\rho\le b\\ \dfrac{I_0}{2\pi\rho}\left(1-\dfrac{\rho^2-b^2}{t^2+2bt}\right)\hat{a}_\phi & b\le\rho\le b+t\\ 0 & \rho\ge b+t\end{cases}$$



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Remaining Steps…

Step 4(alternate) – Calculate total magnetic energy W_m.

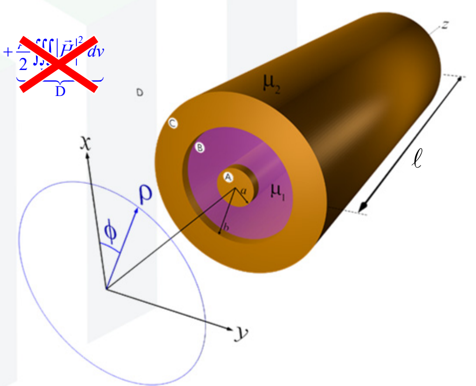
The total magnetic energy is

$$W_\mathrm{m}=\frac{1}{2}\iiint_V \mu\left|\vec{H}\right|^2 dv$$

$$=\underbrace{\frac{\mu}{2}\iiint_V\left|\vec{H}\right|^2 dv}_{A}+\underbrace{\frac{\mu}{2}\iiint_V\left|\vec{H}\right|^2 dv}_{B}+\underbrace{\frac{\mu}{2}\iiint\left|\vec{H}\right|^2 dv}_{C}+\underbrace{\frac{\mu}{2}\iiint\left|\vec{H}\right|^2 dv}_{D}$$

There is no magnetic field outside of the coax.

If the outer conductor is very thin, we can ignore the magnetic energy here.



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Remaining Steps…

Step 4(alternate) – Calculate total magnetic energy W_m.

$$W_m = \underbrace{\frac{\mu}{2}\iiint_V \left|\vec{H}\right|^2 dv}_{A} + \underbrace{\frac{\mu}{2}\iiint_V \left|\vec{H}\right|^2 dv}_{B}$$

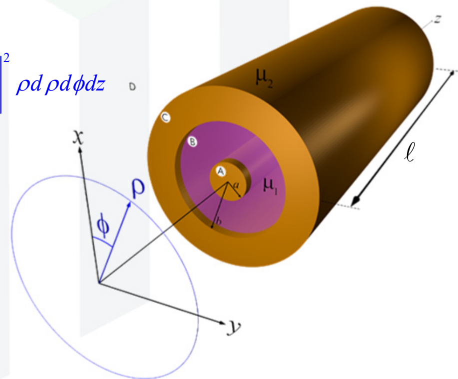
The first term is

$$W_A = \underbrace{\frac{\mu}{2}\iiint_V \left|\vec{H}\right|^2 dv}_{0\le\rho\le a} = \frac{\mu}{2}\int_{z=0}^{\ell}\int_{\phi=0}^{2\pi}\int_{\rho=0}^{a}\left|\frac{I_0\rho}{2\pi a^2}\hat{a}_\phi\right|^2 \rho d\rho d\phi dz$$

$$= \frac{\mu I_0^2}{8\pi^2 a^4}\int_{z=0}^{\ell}\int_{\phi=0}^{2\pi}\int_{\rho=0}^{a}\rho^3 d\rho d\phi dz$$

$$= \frac{\mu I_0^2}{8\pi^2 a^4}\int_{\rho=0}^{a}\left(\int_{\phi=0}^{2\pi} d\phi\right)\left(\int_{z=0}^{\ell} dz\right)\rho^3 d\rho$$

$$= \frac{\mu I_0^2}{8\pi^2 a^4}\int_{\rho=0}^{a}(2\pi)(\ell)\rho^3 d\rho$$



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Remaining Steps…

Step 4(alternate) – Calculate total magnetic energy W_m.

$$W_m = \underbrace{\frac{\mu}{2}\iiint_V \left|\vec{H}\right|^2 dv}_{A} + \underbrace{\frac{\mu}{2}\iiint_V \left|\vec{H}\right|^2 dv}_{B}$$

The first term continued…

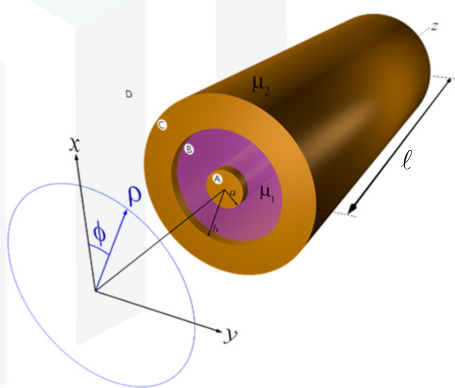
$$W_A = \frac{\mu I_0^2}{8\pi^2 a^4}\int_{\rho=0}^{a}(2\pi)(\ell)\rho^3 d\rho$$

$$= \frac{\mu I_0^2 \ell}{4\pi a^4}\int_{\rho=0}^{a}\rho^3 d\rho$$

$$= \frac{\mu I_0^2 \ell}{4\pi a^4}\left(\frac{\rho^4}{4}\bigg|_0^a\right)$$

$$= \frac{\mu I_0^2 \ell}{4\pi a^4}\left(\frac{a^4}{4} - \frac{0}{4}\right)$$

$$= \frac{\mu I_0^2 \ell}{16\pi}$$



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Remaining Steps…

Step 4(alternate) – Calculate total magnetic energy W_m.

$$W_m = \underbrace{\frac{\mu}{2}\iiint_V \left|\vec{H}\right|^2 dv}_{A} + \underbrace{\frac{\mu}{2}\iiint_V \left|\vec{H}\right|^2 dv}_{B}$$

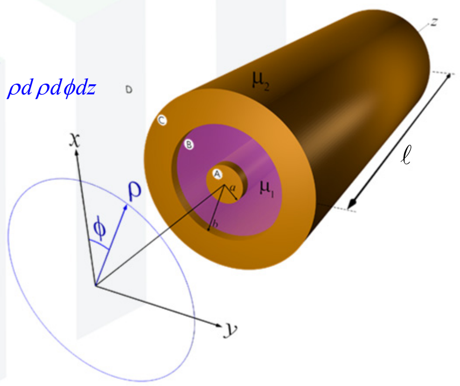
The second term is

$$W_B = \underbrace{\frac{\mu}{2}\iiint_V \left|\vec{H}\right|^2 dv}_{a<\rho\le b} = \frac{\mu}{2}\int_{z=0}^{\ell}\int_{\phi=0}^{2\pi}\int_{\rho=a}^{b}\left|\frac{I_0}{2\pi\rho}\hat{a}_\phi\right|^2 \rho d\rho d\phi dz$$

$$= \frac{\mu I_0^2}{8\pi^2}\int_{z=0}^{\ell}\int_{\phi=0}^{2\pi}\int_{\rho=a}^{b}\frac{1}{\rho}d\rho d\phi dz$$

$$= \frac{\mu I_0^2}{8\pi^2}\int_{\rho=a}^{b}\left(\int_{\phi=0}^{2\pi}d\phi\right)\left(\int_{z=0}^{\ell}dz\right)\frac{1}{\rho}d\rho$$

$$= \frac{\mu I_0^2}{8\pi^2}\int_{\rho=a}^{b}(2\pi)(\ell)\frac{1}{\rho}d\rho$$



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Remaining Steps…

Step 4(alternate) – Calculate total magnetic energy W_m.

$$W_m = \underbrace{\frac{\mu}{2}\iiint_V \left|\vec{H}\right|^2 dv}_{A} + \underbrace{\frac{\mu}{2}\iiint_V \left|\vec{H}\right|^2 dv}_{B}$$

The second term continued…

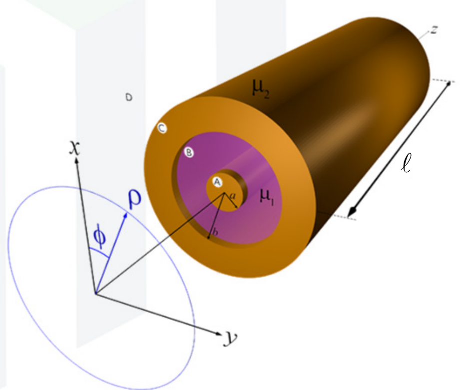
$$W_B = \frac{\mu I_0^2}{8\pi^2}\int_{\rho=a}^{b}(2\pi)(\ell)\frac{1}{\rho}d\rho$$

$$= \frac{\mu I_0^2 \ell}{4\pi}\int_{\rho=a}^{b}\frac{1}{\rho}d\rho$$

$$= \frac{\mu I_0^2 \ell}{4\pi}\left(\ln\rho\big|_a^b\right)$$

$$= \frac{\mu I_0^2 \ell}{4\pi}(\ln b - \ln a)$$

$$= \frac{\mu I_0^2 \ell}{4\pi}\ln\left(\frac{b}{a}\right)$$



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Remaining Steps…

Step 4(alternate) – Calculate total magnetic energy W_m.

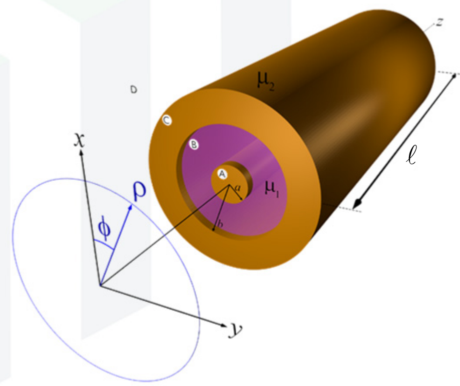
$$W_\mathrm{m} = \underbrace{\frac{\mu}{2}\iiint_V \left|\vec{H}\right|^2 dv}_{\mathrm{A}} + \underbrace{\frac{\mu}{2}\iiint_V \left|\vec{H}\right|^2 dv}_{\mathrm{B}}$$

The total magnetic energy in the coax is

$$W_\mathrm{m} = W_\mathrm{A} + W_\mathrm{B}$$

$$= \frac{\mu I_0^2 \ell}{16\pi} + \frac{\mu I_0^2 \ell}{4\pi} \ln\left(\frac{b}{a}\right)$$

$$= \frac{\mu I_0^2 \ell}{4\pi}\left[\frac{1}{4} + \ln\left(\frac{b}{a}\right)\right]$$



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Remaining Steps…

Step 5(alternate) – Calculate inductance L.

The inductance is

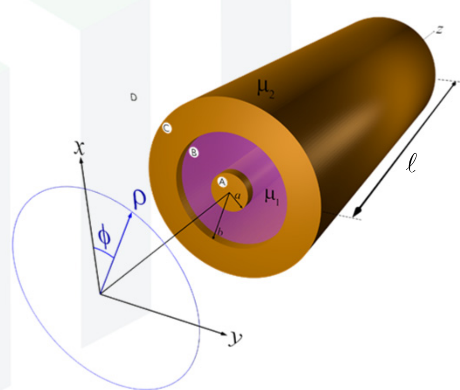
$$L = \frac{2W_\mathrm{m}}{I_0^2}$$

$$= \frac{2\dfrac{\mu I_0^2 \ell}{4\pi}\left[\dfrac{1}{4} + \ln\left(\dfrac{b}{a}\right)\right]}{I_0^2}$$

$$\boxed{L = \frac{\mu \ell}{2\pi}\left[\frac{1}{4} + \ln\left(\frac{b}{a}\right)\right]}$$

The inductance per unit length is

$$\boxed{\frac{L}{\ell} = \frac{\mu}{2\pi}\left[\frac{1}{4} + \ln\left(\frac{b}{a}\right)\right]}$$



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