Reading

Textbooks: *Numerical Methods for Engineers*, 7th Ed.
Steven C. Chapra & Raymond P. Canale
McGraw Hill

Electromagnetic & Photonic Simulation for the Beginner: Finite-Difference Frequency-Domain in MATLAB
Raymond C. Rumpf
Artech House

Website: https://empossible.net/academics/emp4301_5301/

Assignment: Chapra – Read Chapters 29 & 30
Rumpf – Read Chapter 3, pp. 85-94
Website – Topic 7, Lectures 7a-7g

Solve Some 1D Differential Equations

Solve the following ordinary differential equations using the one-dimensional finite-difference method taught in this course and the function `fdder1d()` from Homework 8. Show the formulation of the matrix equation and generate a professional plot of the solution.

Problem #1

$$-3.2\frac{d^2 f(x)}{dx^2} + 7.1\frac{df(x)}{dx} + 2.9f(x) = 0 \qquad -5 \le x \le 5 \qquad \begin{aligned} f(-5) &= -15 \\ f(+5) &= 25 \end{aligned}$$

Problem #2

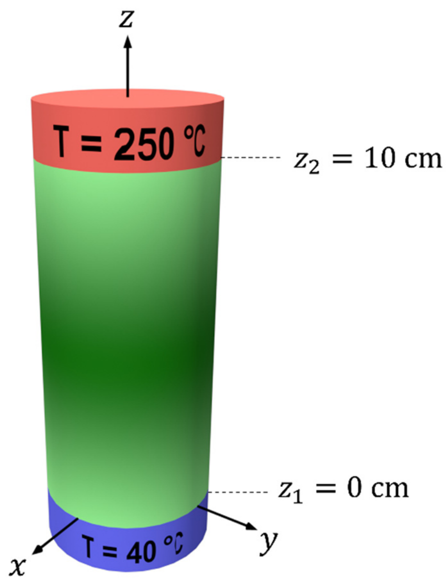
$$\frac{d^2 f(x)}{dx^2} - \frac{df(x)}{dx} - 0.5\sin\left[\pi\frac{x+10}{30}\right]f(x) = 0 \qquad -10 \le x \le 20 \qquad \begin{aligned} f(-10) &= 1 \\ f(+20) &= 2 \end{aligned}$$

Heat Equation

In this homework, you will solve the steady-state heat equation to calculate the temperature along the length of a ceramic rod that has a nonuniform thermal diffusivity α given below.

$$\alpha(z) = \left(2.0\ \text{m}^2/\text{s}\right)\sin\left(\frac{\pi z}{10\ \text{cm}}\right) \tag{1}$$

A diagram of the problem is provided below.



Using two heat plates, the bottom part of the rod is held fixed at 40°C and the top part of the rod is held fixed at 250°C. The rod extends along the z-axis and has a length of 10 cm. The ceramic rod has a nonuniform thermal diffusivity $\alpha(z)$ quantified by Eq. (1).

This homework will step you through the three major phases for numerically solving this type of problem. First is the *formulation* step where all the necessary equations are derived for solving the problem. Second is the *numerical implementation* step where the equations derived in the first step are solved numerically. In this homework, the finite-difference method will be implemented in MATLAB to solve the final differential equation. Last is the *visualization* step where the results of the problem are visualized in a professional manner so that meaningful conclusions can be made about the problem.

Problem #3: Formulation

For this problem, you will derive all the necessary equations needed to implement the solution in MATLAB. All of this should be done by hand or in a word processor. MATLAB should not be used at all. Create a professional quality formulation document that looks like you are explaining and formulating this problem for your own work.

Do not copy/paste anything from this assignment or the internet into your document. Create everything on your own, including diagrams, equations, and text descriptions.

Part 1 – Derive Governing Differential Equation

The general heat equation for a non-uniform isotropic medium is

$$\rho c_{\mathrm{p}} \frac{\partial T(\vec{r},t)}{\partial t} - \nabla \bullet \left[k(\vec{r}) \nabla T(\vec{r},t) \right] = \dot{q}_{\mathrm{V}} \tag{2}$$

In this equation $T(\vec{r},t)$ is the temperature as a function of position $\vec{r}$ and time t, ρ is the mass density, c_{p} is the specific heat capacity, k is the thermal conductivity, and $\dot{q}_{\mathrm{V}}$ describes volumetric heating. This equation is simplified in two ways. First, the ceramic rod is not generating heat on its own so $\dot{q}_{\mathrm{V}} = 0$. All heat sources are external to the rod. Second, if the equation is divided by ρc_{p}, the term $k(\vec{r})/\rho c_{\mathrm{p}}$ is called the thermal diffusivity and is written as the single function $\alpha(\vec{r})$. Thermal diffusivity measures the rate of transfer of heat from hot to cold. Applying these two simplifications to Eq. (2) and rearranging terms gives

$$\frac{\partial T(\vec{r},t)}{\partial t} = \nabla \bullet \left[\alpha(\vec{r}) \nabla T(\vec{r},t) \right] \tag{3}$$

The temperature has units of °C and thermal diffusivity has units of m²/s. To obtain the steady-state solution, the time derivative is set to zero and the equation reduces to

$$\nabla \bullet \left[\alpha(\vec{r}) \nabla T(\vec{r},t) \right] = 0 \tag{4}$$

Starting from Eq. (2), derive Eq. (4) in detailed steps. Describe the equations, variables, and their units.

Part 2 – Reduce to One-Dimension

It is excellent practice to reduce this to a one-dimensional problem. This is valid as long as the problem is approximately uniform in two directions. Let the two uniform directions be x and y. Under this assumption, Eq. (4) reduces to

$$\frac{d}{dz}\left[\alpha(z) \frac{dT(z,t)}{dz} \right] = 0 \tag{5}$$

Applying the derivative operation gives

$$\frac{d^2T(z,t)}{dz^2}+\left(\frac{1}{\alpha(z)}\frac{d\alpha(z)}{dz}\right)\frac{dT(z,t)}{dz}=0 \tag{6}$$

Add explanation and fill in all of the vector calculus steps to derive Eq. (6) from Eq. (4). Recall that the del operator is a vector defined as $\nabla = (\partial/\partial x)\hat{a}_x + (\partial/\partial y)\hat{a}_y + (\partial/\partial z)\hat{a}_z$.

Part 3 – Derive Matrix Equation

To solve Eq. (5) using the finite-difference method, the temperature $T(z)$ and thermal diffusivity $\alpha(z)$ are made discrete functions along a one-dimensional grid representing the z-axis. The matrix equation derived from Eq. (6) has the following form.

$$\left[D_z^2\right][T]+[C][D_z][T]=[0] \tag{7}$$

This equation is then put into standard form as

$$[A][T]=[0] \tag{8}$$

$$[A]=\left[D_z^2\right]+[C][D_z] \tag{9}$$

Fill in all of the steps to derive Eqs. (8) and (9) from Eq. (6). You should not have to explicitly handle any finite-differences to do this. Your new superpowers should make this step almost effortless, but add explanations and attempt to visualize the step as best as possible. Visualize all the matrices and column vectors in Eq. (7). For example, $[D_z^2]$ might be visualized as

$$\left[D_z^2\right]=\frac{1}{(\Delta z)^2}\begin{bmatrix} -2 & 1 & & & & & \\ 1 & -2 & 1 & & & & \\ & 1 & -2 & 1 & & & \\ & & 1 & -2 & 1 & & \\ & & & & \ddots & & \\ & & & & 1 & -2 & 1 \\ & & & & & 1 & -2 \end{bmatrix} \tag{10}$$

Part 5 – Apply Boundary Conditions

Equation (7) is not yet solvable because $[T] = [D_z^2]^{-1}[0] = [0]$ is a trivial solution. It must be put into the form $[A'][T] = [b]$ by applying the boundary conditions.

$$\underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ a_{21} & a_{22} & \cdots & a_{2,N-1} & a_{2N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{N-1,1} & a_{N-1,2} & \cdots & a_{N-1,N-1} & a_{N-1,N} \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}}_{[A']}\underbrace{\begin{bmatrix} T(1) \\ T(2) \\ \vdots \\ T(N-1) \\ T(N) \end{bmatrix}}_{[T]}=\underbrace{\begin{bmatrix} 40°C \\ 0 \\ \vdots \\ 0 \\ 250°C \end{bmatrix}}_{[b]}\begin{matrix} \rightarrow T(z_1)=40°C \\ \\ \\ \\ \rightarrow T(z_2)=250°C \end{matrix} \tag{11}$$

Fill in the steps to derive this result from Eq. (7). Describe the operations that must be performed in order to incorporate the boundary conditions. Visualize and explain this step as best as possible.

Part 6 – Solve for $[T]$

Complete your formulation by stating that the temperature profile $T(z)$ along the bar is calculated from Eq. (11) as follows.

$$[T]=[A']^{-1}[b] \quad (12)$$

Problem #4: Implementation

In this problem, you will implement the finite-difference method that you formulated in Problem #3. Your implementation should have very clean MATLAB code that is well-organized and well-commented. At the end, generate a crude plot of the temperature profile $T(z)$ obtained from your program. Do not worry about quality graphics yet. That will be addressed in Problem #5. Your program should follow the steps outlined below.

Part 1 – Program Header and Dashboard

Start your MATLAB program with the following header:

```
% HW9_Prob4.m
%
% Homework #9, Problem #2
% Computational Methods
% HW9_Prob4.m
%
% Homework #9, Problem #2
% Computational Methods
% Instructor: Dr. Raymond C. Rumpf

% INITIALIZE MATLAB
close all;
clc;
clear all;

% UNITS
centimeters = 0.01;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% DASHBOARD
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% BOUNDARY CONDITIONS
z1 =  0 * centimeters;       T1 =  40;
z2 = 10 * centimeters;       T2 = 250;

% NUMBER OF GRID POINTS
Nz = 100;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% PERFFORM FINITE-DIFFERENCE METHOD
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

Part 2 – Calculate the Grid

Calculate Δz and calculate a one-dimensional array of points `z` describing the position of each of the points in the interval $z_1 \leq z \leq z_2$.

Part 3 – Build the Derivative Matrix

Construct the derivative matrix `DZ2` by calling your `fdder1d()` function that you wrote in Homework #8.

Part 4 – Build Matrix $[A]$

Build the matrix `A` you formulated in Eq. (7).

Part 5 – Initialize Column Vector $[b]$

Initialize a column vector `b` to contain all zeros.

Part 6 – Apply Boundary Condition at $z = z_1$

Modify the matrix `A` and the column vector `b` consistent with Eq. (11) to incorporate the boundary condition at $z = z_1$.

Part 7 – Apply Boundary Condition at $z = z_2$

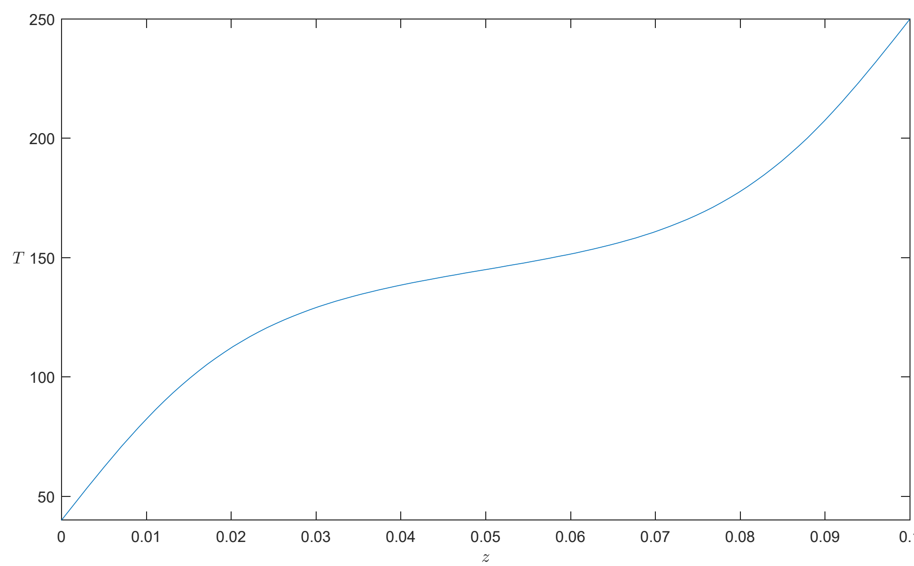
Modify the matrix `A` and the column vector `b` consistent with Eq. (11) to incorporate the boundary condition at $z = z_2$.

Part 8 – Calculate the Temperature Column Vector $[T]$

Calculate the temperature profile function `T` by solving Eq. (12).

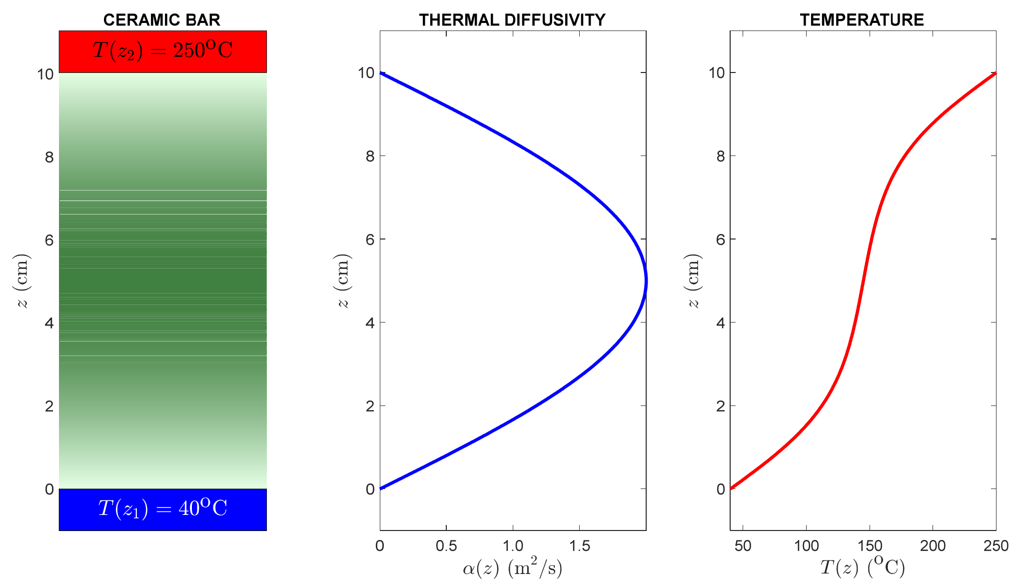
Part 9 – Plot $T(z)$ in the Interval $z_1 \leq z \leq z_2$

Plot the temperature profile using just `plot(z,T)`. Use the default MATLAB graphics for this. You will make it look professional in the next problem so no need to do that here. Your plot should look like what is shown below.



Problem #5: Visualization

In this problem, you will generate professional quality graphics to visualize your problem and its solution following a series of steps. Your figure will contain three subplots placed horizontally next to each other: (1) the device, (2) the thermal diffusivity function $\alpha(z)$, and (3) the temperature profile $T(z)$. In the end, your figure and results should look something like below. Try to beat this level of clarity and professionalism in your homework.



Part 1 – Open a New Figure Window

Open a new figure window that has a white background. Size the window such that the y-axis of all the different subplots line up and match the device.

Part 2 – Open a First Subplot to Show the Device

Select the first subplot using the `subplot()` function in MATLAB.

Part 3 – Define the RGB Values for Light and Dark Green

The values used in the solution for this homework were:

```
c1 = [0.90 1.0 0.90];
c2 = [0.25 0.5 0.25];
```

Feel free to pick your favorite!

Part 4 – Draw the Bottom Temperature

Use the `fill()` command to draw a rectangle below your ceramic rod. Fill the rectangle with blue and add a text label with the temperature.

Part 5 – Draw the Top Temperature

Use the `fill()` command to draw a rectangle above your ceramic rod. Fill the rectangle with red and add a text label with the temperature.

Part 6 – Draw the Ceramic to Convey Thermal Diffusivity

Use a loop to draw the ceramic rod one slice (thin rectangle) at a time along the z-axis. Use the `fill()` command to visualize each ceramic slice with a green color. Do not use an edge line for the rectangles. Scale its shade of green to convey the magnitude of the thermal diffusivity $\alpha(z)$, where the darker shade corresponds to higher thermal diffusivity. Consider calculating the colors using code something like (i.e. may require some modification)

```
c  = (1 - alpha(nz))*c1 + alpha(nz)*c2;
```

Part 7 – Set the Graphics View for the First Subplot

Set a professional view for the device graphic by turning off the axes, giving the subplot the title "CERAMIC BAR," and perhaps other options of your choosing.

Part 8 – Calculate the Vertical Tick Positions and Labels for the Remaining Plots

Your tick marks should go from 0 cm to 10 cm in steps of 2.0 cm. Your labels should all have the same number of digits, except for zero which should only have one digit.

Part 9 – Plot the Thermal Diffusivity Function $\alpha(z)$

In the first subplot from the right of the device, plot the thermal diffusivity function $\alpha(z)$ using a dark blue line. The units for the vertical axis should be centimeters. Be sure linewidths are sufficient, but not overbearing. Label the x-axis with "$\alpha(z)$ (mm^2/s)" and label the y-axis with "z (cm)." Set the y-axis limits so that the position of the function corresponds to the position of the devices to its left. Give the subplot the title "THERMAL DIFFUSIVITY." Dress up anything else in the plot that does not look professional and ready for publication.

Part 10 – Plot the Temperature Profile $T(z)$

In the last subplot on the right, plot the calculated temperature profile $T(z)$ using a red line. The units for the vertical axis should be centimeters. Be sure linewidths are sufficient, but not overbearing. Label the x-axis with "$T(z)$ (°C)" and label the y-axis with "z (cm)." Set the y-axis limits so that the position of the function corresponds to the position of the devices to its left. Give the subplot the title "TEMPERATURE." Dress up anything else in the plot that does not look professional and ready for publication.