Electromagnetics:
Electromagnetic Field Theory

Equations for LHI Waveguide Analysis

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Starting Point

Recall the starting point for waveguide analysis.

$$\frac{\partial E_{0,z}}{\partial y} + j\beta E_{0,y} = -j\omega\mu H_{0,x}$$

$$-j\beta E_{0,x} - \frac{\partial E_{0,z}}{\partial x} = -j\omega\mu H_{0,y}$$

$$\frac{\partial E_{0,y}}{\partial x} - \frac{\partial E_{0,x}}{\partial y} = -j\omega\mu H_{0,z}$$

$$\frac{\partial H_{0,z}}{\partial y} + j\beta H_{0,y} = j\omega\varepsilon E_{0,x}$$

$$-j\beta H_{0,x} - \frac{\partial H_{0,z}}{\partial x} = j\omega\varepsilon E_{0,y}$$

$$\frac{\partial H_{0,y}}{\partial x} - \frac{\partial H_{0,x}}{\partial y} = j\omega\varepsilon E_{0,z}$$

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Reducing Number of Terms

It is possible to put $E_{0,x}$, $E_{0,y}$, $H_{0,x}$, and $H_{0,y}$ in terms of just $E_{0,z}$ and $H_{0,z}$.

$$\frac{\partial E_{0,z}}{\partial y} + j\beta E_{0,y} = -j\omega\mu H_{0,x}$$

$$-j\beta E_{0,x} - \frac{\partial E_{0,z}}{\partial x} = -j\omega\mu H_{0,y}$$

$$\frac{\partial E_{0,y}}{\partial x} - \frac{\partial E_{0,x}}{\partial y} = -j\omega\mu H_{0,z}$$

$$\frac{\partial H_{0,z}}{\partial y} + j\beta H_{0,y} = j\omega\varepsilon E_{0,x}$$

$$-j\beta H_{0,x} - \frac{\partial H_{0,z}}{\partial x} = j\omega\varepsilon E_{0,y}$$

$$\frac{\partial H_{0,y}}{\partial x} - \frac{\partial H_{0,x}}{\partial y} = j\omega\varepsilon E_{0,z}$$

$$\Rightarrow$$

$$E_{0,x} = -\frac{j}{k^2-\beta^2}\left(\beta\frac{\partial E_{0,z}}{\partial x} + \omega\mu\frac{\partial H_{0,z}}{\partial y}\right)$$

$$E_{0,y} = \frac{j}{k^2-\beta^2}\left(-\beta\frac{\partial E_{0,z}}{\partial y} + \omega\mu\frac{\partial H_{0,z}}{\partial x}\right)$$

$$H_{0,x} = \frac{j}{k^2-\beta^2}\left(\omega\varepsilon\frac{\partial E_{0,z}}{\partial y} - \beta\frac{\partial H_{0,z}}{\partial x}\right)$$

$$H_{0,y} = -\frac{j}{k^2-\beta^2}\left(\omega\varepsilon\frac{\partial E_{0,z}}{\partial x} + \beta\frac{\partial H_{0,z}}{\partial y}\right)$$

To analyze a waveguide, it is only necessary to solve for $E_{0,z}$ and $H_{0,z}$.

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Derivation (1 of 2)

$$\frac{\partial E_{0,z}}{\partial y} + j\beta E_{0,y} = -j\omega\mu H_{0,x} \quad \text{Eq. (1a)}$$

$$-j\beta E_{0,x} - \frac{\partial E_{0,z}}{\partial x} = -j\omega\mu H_{0,y} \quad \text{Eq. (1b)}$$

$$\frac{\partial E_{0,y}}{\partial x} - \frac{\partial E_{0,x}}{\partial y} = -j\omega\mu H_{0,z} \quad \text{Eq. (1c)}$$

$$\frac{\partial H_{0,z}}{\partial y} + j\beta H_{0,y} = j\omega\varepsilon E_{0,x} \quad \text{Eq. (1d)}$$

$$-j\beta H_{0,x} - \frac{\partial H_{0,z}}{\partial x} = j\omega\varepsilon E_{0,y} \quad \text{Eq. (1e)}$$

$$\frac{\partial H_{0,y}}{\partial x} - \frac{\partial H_{0,x}}{\partial y} = j\omega\varepsilon E_{0,z} \quad \text{Eq. (1f)}$$

Step 1 – Solve Eq. (1e) for $E_{0,y}$.

$$E_{0,y} = \frac{1}{j\omega\varepsilon}\left(-j\beta H_{0,x} - \frac{\partial H_{0,z}}{\partial x}\right)$$

Step 2 – Substitute this expression into Eq. (1a) to eliminate $E_{0,y}$.

$$\frac{\partial E_{0,z}}{\partial y} + j\beta\left[\frac{1}{j\omega\varepsilon}\left(-j\beta H_{0,x} - \frac{\partial H_{0,z}}{\partial x}\right)\right] = -j\omega\mu H_{0,x}$$

Step 3 – Recall that $k^2 = \omega^2\mu\varepsilon$ and solve this new expression for $H_{0,x}$.

$$H_{0,x} = \frac{j}{k^2-\beta^2}\left(\omega\varepsilon\frac{\partial E_{0,z}}{\partial y} - \beta\frac{\partial H_{0,z}}{\partial x}\right)$$

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Derivation (2 of 2)

Step 4 – Derive three more similar equations.

Solve Eq. (1d) for $E_{0,x}$, substitute that expression into Eq. (1b) and solve for $H_{0,y}$.

$$H_{0,y} = -\frac{j}{k^2 - \beta^2}\left(\omega\varepsilon\frac{\partial E_{0,z}}{\partial x} + \beta\frac{\partial H_{0,z}}{\partial y}\right)$$

Solve Eq. (1b) for $H_{0,y}$, substitute that expression into Eq. (1d) and solve for $E_{0,x}$.

$$E_{0,x} = -\frac{j}{k^2 - \beta^2}\left(\beta\frac{\partial E_{0,z}}{\partial x} + \omega\mu\frac{\partial H_{0,z}}{\partial y}\right)$$

Solve Eq. (1a) for $H_{0,x}$, substitute that expression into Eq. (1e) and solve for $E_{0,y}$.

$$E_{0,y} = \frac{j}{k^2 - \beta^2}\left(-\beta\frac{\partial E_{0,z}}{\partial y} + \omega\mu\frac{\partial H_{0,z}}{\partial x}\right)$$

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Final Form of Reduced Set of Equations

Step 5 – Define the *cutoff wave number* k_c as

$$k_c^2 = k^2 - \beta^2$$

This term will have more meaning later.

Now all of the transverse field components $E_{0,x}$, $E_{0,y}$, $H_{0,x}$ and $H_{0,y}$ are expressed in terms of just the two longitudinal components $E_{0,z}$ and $H_{0,z}$.

$$H_{0,x} = \frac{j}{k_c^2}\left(\omega\varepsilon\frac{\partial E_{0,z}}{\partial y} - \beta\frac{\partial H_{0,z}}{\partial x}\right) \qquad E_{0,x} = -\frac{j}{k_c^2}\left(\beta\frac{\partial E_{0,z}}{\partial x} + \omega\mu\frac{\partial H_{0,z}}{\partial y}\right)$$

$$H_{0,y} = -\frac{j}{k_c^2}\left(\omega\varepsilon\frac{\partial E_{0,z}}{\partial x} + \beta\frac{\partial H_{0,z}}{\partial y}\right) \qquad E_{0,y} = \frac{j}{k_c^2}\left(-\beta\frac{\partial E_{0,z}}{\partial y} + \omega\mu\frac{\partial H_{0,z}}{\partial x}\right)$$

Analyzing a waveguide reduces to just solving for $E_{0,z}$ and $H_{0,z}$. The remaining field components can be calculated directly from these two terms.

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How To Find $E_{0,z}$ and $H_{0,z}$?

Recall that in LHI media, the wave equation simplified to

$$\nabla\times\left(\mu^{-1}\nabla\times\vec{E}\right)=\omega^2\varepsilon\vec{E} \qquad \nabla\times\left(\varepsilon^{-1}\nabla\times\vec{H}\right)=\omega^2\mu\vec{H}$$

$$\downarrow \qquad \downarrow$$

$$\nabla^2\vec{E}+k^2\vec{E}=0 \qquad \nabla^2\vec{H}+k^2\vec{H}=0$$

$$\downarrow \qquad \downarrow$$

$$\nabla^2E_x+k^2E_x=0 \qquad \nabla^2H_x+k^2H_x=0$$

$$\nabla^2E_y+k^2E_y=0 \qquad \nabla^2H_y+k^2H_y=0$$

$$\nabla^2E_z+k^2E_z=0 \qquad \nabla^2H_z+k^2H_z=0$$

Substituting the solution $E_z = E_{0,z}e^{-j\beta z}$ and $H_z = H_{0,z}e^{-j\beta z}$ into the bottom equations above gives

$$\boxed{\nabla^2E_{0,z}+k_c^2E_{0,z}=0} \qquad \boxed{\nabla^2H_{0,z}+k_c^2H_{0,z}=0}$$

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Summary of Equations for LHI Waveguide Analysis

Step 1 – Solve for $E_{0,z}$ and $H_{0,z}$

For LHI waveguides

$$\nabla^2E_{0,z}+k_c^2E_{0,z}=0 \quad \text{TM Modes}$$

$$\nabla^2H_{0,z}+k_c^2H_{0,z}=0 \quad \text{TE Modes}$$

Step 2 – Calculate $E_{0,x}$, $E_{0,y}$, $H_{0,x}$ and $H_{0,y}$

$$E_{0,x}=-\frac{j}{k^2-\beta^2}\left(\beta\frac{\partial E_{0,z}}{\partial x}+\omega\mu\frac{\partial H_{0,z}}{\partial y}\right)$$

$$E_{0,y}=\frac{j}{k^2-\beta^2}\left(-\beta\frac{\partial E_{0,z}}{\partial y}+\omega\mu\frac{\partial H_{0,z}}{\partial x}\right)$$

$$H_{0,x}=\frac{j}{k^2-\beta^2}\left(\omega\varepsilon\frac{\partial E_{0,z}}{\partial y}-\beta\frac{\partial H_{0,z}}{\partial x}\right)$$

$$H_{0,y}=-\frac{j}{k^2-\beta^2}\left(\omega\varepsilon\frac{\partial E_{0,z}}{\partial x}+\beta\frac{\partial H_{0,z}}{\partial y}\right)$$

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Solution Categories

Hybrid Mode Analysis

$E_{0,z} \neq 0$ and $H_{0,z} \neq 0$

Can analyze any waveguide.
Usually very complicated.

- No simplifications to the analysis are possible.
- Solutions contain all six field components

TEM Analysis

$E_{0,z} = H_{0,z} = 0$

- Must be a transmission line,
- and must have a homogeneous dielectric

- **Transverse electromagnetic (TEM) mode**
- Both electric and magnetic fields are transverse to the direction of propagation.

TE & TM Analysis

$E_{0,z} = 0$ or $H_{0,z} = 0$, but not both

- Homogeneous dielectric,
- or has a dielectric uniform in one direction

- **Transverse electric (TE) mode**
- $E_{0,z} = 0$ so the electric field is completely transverse to the direction of propagation.

- **Transverse magnetic (TM) mode**
- $H_{0,z} = 0$ so the magnetic field is completely transverse to the direction of propagation.

Slab Waveguide Analysis

- Material properties uniform and infinite along one direction

- Maxwell's equations simplify and decouple into two independent sets of modes.

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