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Electromagnetics:
Electromagnetic Field Theory

Full Wave Analysis Setup for Waveguides

1

Solution Categories

Hybrid Mode Analysis

$E_{0,z} \neq 0$ and $H_{0,z} \neq 0$

Can analyze any waveguide.
Usually very complicated.

- No simplifications to the analysis are possible.
- Solutions contain all six field components

TEM Analysis

$E_{0,z} = H_{0,z} = 0$

- Must be a transmission line,
- and must have a homogeneous dielectric

- Transverse electromagnetic (TEM) mode
- Both electric and magnetic fields are transverse to the direction of propagation.

TE & TM Analysis

$E_{0,z} = 0$ or $H_{0,z} = 0$, but not both

- Homogeneous dielectric,
- or has a dielectric uniform in one direction

- Transverse electric (TE) mode
- $E_{0,z} = 0$ so the electric field is completely transverse to the direction of propagation.
- Transverse magnetic (TM) mode
- $H_{0,z} = 0$ so the magnetic field is completely transverse to the direction of propagation.

Slab Waveguide Analysis

- Material properties uniform and infinite along one direction

- Maxwell's equations simplify and decouple into two independent sets of modes.

Slide 2

2

Eliminate $E_{0,z}$ and $H_{0,z}$

For full wave analysis, back up to Maxwell's equations in linear and isotropic media (i.e. can be inhomogeneous).

$$\frac{\partial E_{0,z}}{\partial y} + j\beta E_{0,y} = -j\omega\mu H_{0,x} \quad \text{Eq. (1a)}$$

$$-j\beta E_{0,x} - \frac{\partial E_{0,z}}{\partial x} = -j\omega\mu H_{0,y} \quad \text{Eq. (1b)}$$

$$\frac{\partial E_{0,y}}{\partial x} - \frac{\partial E_{0,x}}{\partial y} = -j\omega\mu H_{0,z} \quad \text{Eq. (1c)}$$

$$\frac{\partial H_{0,z}}{\partial y} + j\beta H_{0,y} = j\omega\varepsilon E_{0,x} \quad \text{Eq. (2a)}$$

$$-j\beta H_{0,x} - \frac{\partial H_{0,z}}{\partial x} = j\omega\varepsilon E_{0,y} \quad \text{Eq. (2b)}$$

$$\frac{\partial H_{0,y}}{\partial x} - \frac{\partial H_{0,x}}{\partial y} = j\omega\varepsilon E_{0,z} \quad \text{Eq. (2c)}$$

Solve Eq. (1c) for $H_{0,z}$ and solve Eq. (2c) for $E_{0,z}$.

$$H_{0,z} = -\frac{1}{j\omega\mu}\left(\frac{\partial E_{0,y}}{\partial x} - \frac{\partial E_{0,x}}{\partial y}\right) \quad \text{Eq. (3a)}$$

$$E_{0,z} = \frac{1}{j\omega\varepsilon}\left(\frac{\partial H_{0,y}}{\partial x} - \frac{\partial H_{0,x}}{\partial y}\right) \quad \text{Eq. (3b)}$$

Substitute Eq. (3a) into Eqs. (2a) and (2b), & substitute Eq. (3b) into Eqs. (1a) and (1b).

$$\frac{1}{\omega}\frac{\partial}{\partial x}\left[\frac{1}{\varepsilon}\left(\frac{\partial H_{0,y}}{\partial x} - \frac{\partial H_{0,x}}{\partial y}\right)\right] + \omega\mu H_{0,y} = \beta E_{0,x} \quad \text{Eq. (4a)}$$

$$\frac{1}{\omega}\frac{\partial}{\partial y}\left[\frac{1}{\varepsilon}\left(\frac{\partial H_{0,y}}{\partial x} - \frac{\partial H_{0,x}}{\partial y}\right)\right] + \omega\mu H_{0,x} = \beta E_{0,y} \quad \text{Eq. (4b)}$$

$$-\frac{1}{\omega}\frac{\partial}{\partial x}\left[\frac{1}{\mu}\left(\frac{\partial E_{0,y}}{\partial x} - \frac{\partial E_{0,x}}{\partial y}\right)\right] - \omega\varepsilon E_{0,y} = \beta H_{0,x} \quad \text{Eq. (5a)}$$

$$-\frac{1}{\omega}\frac{\partial}{\partial y}\left[\frac{1}{\mu}\left(\frac{\partial E_{0,y}}{\partial x} - \frac{\partial E_{0,x}}{\partial y}\right)\right] - \omega\varepsilon E_{0,x} = \beta H_{0,y} \quad \text{Eq. (5b)}$$

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Form Two Matrix Equations

The four equations can be expressed as two matrix equations.

$$\frac{1}{\omega}\frac{\partial}{\partial x}\left[\frac{1}{\varepsilon}\left(\frac{\partial H_{0,y}}{\partial x} - \frac{\partial H_{0,x}}{\partial y}\right)\right] + \omega\mu H_{0,y} = \beta E_{0,x} \quad \text{Eq. (4a)}$$

$$\frac{1}{\omega}\frac{\partial}{\partial y}\left[\frac{1}{\varepsilon}\left(\frac{\partial H_{0,y}}{\partial x} - \frac{\partial H_{0,x}}{\partial y}\right)\right] + \omega\mu H_{0,x} = \beta E_{0,y} \quad \text{Eq. (4b)}$$

$\Rightarrow$

$$\begin{bmatrix} -\frac{1}{\omega}\frac{\partial}{\partial x}\frac{1}{\varepsilon}\frac{\partial}{\partial y} & \frac{1}{\omega}\frac{\partial}{\partial x}\frac{1}{\varepsilon}\frac{\partial}{\partial x} + \omega\mu \\ -\left(\frac{1}{\omega}\frac{\partial}{\partial y}\frac{1}{\varepsilon}\frac{\partial}{\partial y} + \omega\mu\right) & \frac{1}{\omega}\frac{\partial}{\partial y}\frac{1}{\varepsilon}\frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} H_{0,x} \\ H_{0,y} \end{bmatrix} = \beta \begin{bmatrix} E_{0,x} \\ E_{0,y} \end{bmatrix} \quad \text{Eq. (6)}$$

$$-\frac{1}{\omega}\frac{\partial}{\partial x}\left[\frac{1}{\mu}\left(\frac{\partial E_{0,y}}{\partial x} - \frac{\partial E_{0,x}}{\partial y}\right)\right] - \omega\varepsilon E_{0,y} = \beta H_{0,x} \quad \text{Eq. (5a)}$$

$$-\frac{1}{\omega}\frac{\partial}{\partial y}\left[\frac{1}{\mu}\left(\frac{\partial E_{0,y}}{\partial x} - \frac{\partial E_{0,x}}{\partial y}\right)\right] - \omega\varepsilon E_{0,x} = \beta H_{0,y} \quad \text{Eq. (5b)}$$

$\Rightarrow$

$$\begin{bmatrix} \frac{1}{\omega}\frac{\partial}{\partial x}\frac{1}{\mu}\frac{\partial}{\partial y} & -\left(\frac{1}{\omega}\frac{\partial}{\partial x}\frac{1}{\mu}\frac{\partial}{\partial x} + \omega\varepsilon\right) \\ \frac{1}{\omega}\frac{\partial}{\partial y}\frac{1}{\mu}\frac{\partial}{\partial y} + \omega\varepsilon & -\frac{1}{\omega}\frac{\partial}{\partial y}\frac{1}{\mu}\frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} E_{0,x} \\ E_{0,y} \end{bmatrix} = \beta \begin{bmatrix} H_{0,x} \\ H_{0,y} \end{bmatrix} \quad \text{Eq. (7)}$$

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Form a Single Matrix Equation

The two matrix equations from the previous slide were

$$\begin{bmatrix} -\frac{1}{\omega}\frac{\partial}{\partial x}\frac{1}{\varepsilon}\frac{\partial}{\partial y} & \frac{1}{\omega}\frac{\partial}{\partial x}\frac{1}{\varepsilon}\frac{\partial}{\partial x}+\omega\mu \\ -\left(\frac{1}{\omega}\frac{\partial}{\partial y}\frac{1}{\varepsilon}\frac{\partial}{\partial y}+\omega\mu\right) & \frac{1}{\omega}\frac{\partial}{\partial y}\frac{1}{\varepsilon}\frac{\partial}{\partial x} \end{bmatrix}\begin{bmatrix} H_{0,x} \\ H_{0,y} \end{bmatrix}=\beta\begin{bmatrix} E_{0,x} \\ E_{0,y} \end{bmatrix} \quad \text{Eq. (6)}$$

$$\begin{bmatrix} \frac{1}{\omega}\frac{\partial}{\partial x}\frac{1}{\mu}\frac{\partial}{\partial y} & -\left(\frac{1}{\omega}\frac{\partial}{\partial x}\frac{1}{\mu}\frac{\partial}{\partial x}+\omega\varepsilon\right) \\ \frac{1}{\omega}\frac{\partial}{\partial y}\frac{1}{\mu}\frac{\partial}{\partial y}+\omega\varepsilon & -\frac{1}{\omega}\frac{\partial}{\partial y}\frac{1}{\mu}\frac{\partial}{\partial x} \end{bmatrix}\begin{bmatrix} E_{0,x} \\ E_{0,y} \end{bmatrix}=\beta\begin{bmatrix} H_{0,x} \\ H_{0,y} \end{bmatrix} \quad \text{Eq. (7)}$$

Solve Eq. (7) for the magnetic field components.

$$\begin{bmatrix} H_{0,x} \\ H_{0,y} \end{bmatrix}=\frac{1}{\beta}\begin{bmatrix} \frac{1}{\omega}\frac{\partial}{\partial x}\frac{1}{\mu}\frac{\partial}{\partial y} & -\left(\frac{1}{\omega}\frac{\partial}{\partial x}\frac{1}{\mu}\frac{\partial}{\partial x}+\omega\varepsilon\right) \\ \frac{1}{\omega}\frac{\partial}{\partial y}\frac{1}{\mu}\frac{\partial}{\partial y}+\omega\varepsilon & -\frac{1}{\omega}\frac{\partial}{\partial y}\frac{1}{\mu}\frac{\partial}{\partial x} \end{bmatrix}\begin{bmatrix} E_{0,x} \\ E_{0,y} \end{bmatrix} \quad \text{Eq. (8)}$$

Substitute Eq. (8) into Eq. (6) to eliminate the magnetic field terms and derive a wave equation.

$$\begin{bmatrix} -\frac{1}{\omega}\frac{\partial}{\partial x}\frac{1}{\varepsilon}\frac{\partial}{\partial y} & \frac{1}{\omega}\frac{\partial}{\partial x}\frac{1}{\varepsilon}\frac{\partial}{\partial x}+\omega\mu \\ -\left(\frac{1}{\omega}\frac{\partial}{\partial y}\frac{1}{\varepsilon}\frac{\partial}{\partial y}+\omega\mu\right) & \frac{1}{\omega}\frac{\partial}{\partial y}\frac{1}{\varepsilon}\frac{\partial}{\partial x} \end{bmatrix}\begin{bmatrix} \frac{1}{\omega}\frac{\partial}{\partial x}\frac{1}{\mu}\frac{\partial}{\partial y} & -\left(\frac{1}{\omega}\frac{\partial}{\partial x}\frac{1}{\mu}\frac{\partial}{\partial x}+\omega\varepsilon\right) \\ \frac{1}{\omega}\frac{\partial}{\partial y}\frac{1}{\mu}\frac{\partial}{\partial y}+\omega\varepsilon & -\frac{1}{\omega}\frac{\partial}{\partial y}\frac{1}{\mu}\frac{\partial}{\partial x} \end{bmatrix}\begin{bmatrix} E_{0,x} \\ E_{0,y} \end{bmatrix}-\beta^2\begin{bmatrix} E_{0,x} \\ E_{0,y} \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Yikes!! ☹ This is typically solved numerically on a computer.

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Quasi-LP Analysis

In many cases the hybrid modes are strongly linearly polarized. For these cases, a simplifying approximation can be made that the cross coupling between $E_{0,x}$ and $E_{0,y}$ is weak and can be neglected. Under this condition, the governing equation separates into two independent equations, one for each LP mode.

$$\begin{bmatrix} -\frac{1}{\omega}\frac{\partial}{\partial x}\frac{1}{\varepsilon}\frac{\partial}{\partial y} & \frac{1}{\omega}\frac{\partial}{\partial x}\frac{1}{\varepsilon}\frac{\partial}{\partial x}+\omega\mu \\ -\left(\frac{1}{\omega}\frac{\partial}{\partial y}\frac{1}{\varepsilon}\frac{\partial}{\partial y}+\omega\mu\right) & \frac{1}{\omega}\frac{\partial}{\partial y}\frac{1}{\varepsilon}\frac{\partial}{\partial x} \end{bmatrix}\begin{bmatrix} \frac{1}{\omega}\frac{\partial}{\partial x}\frac{1}{\mu}\frac{\partial}{\partial y} & -\left(\frac{1}{\omega}\frac{\partial}{\partial x}\frac{1}{\mu}\frac{\partial}{\partial x}+\omega\varepsilon\right) \\ \frac{1}{\omega}\frac{\partial}{\partial y}\frac{1}{\mu}\frac{\partial}{\partial y}+\omega\varepsilon & -\frac{1}{\omega}\frac{\partial}{\partial y}\frac{1}{\mu}\frac{\partial}{\partial x} \end{bmatrix}\begin{bmatrix} E_{0,x} \\ E_{0,y} \end{bmatrix}-\beta^2\begin{bmatrix} E_{0,x} \\ E_{0,y} \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

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Quasi-LP Analysis

In many cases the hybrid modes are strongly linearly polarized. For these cases, a simplifying approximation can be made that the cross coupling between $E_{0,x}$ and $E_{0,y}$ is weak and can be neglected. Under this condition, the governing equation separates into two independent equations, one for each LP mode.

$$\begin{bmatrix} \Omega_{xx} & \Omega_{xy} \\ \Omega_{yx} & \Omega_{yy} \end{bmatrix} \begin{bmatrix} E_{0,x} \\ E_{0,y} \end{bmatrix} - \beta^2 \begin{bmatrix} E_{0,x} \\ E_{0,y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Quasi-LP Analysis

In many cases the hybrid modes are strongly linearly polarized. For these cases, a simplifying approximation can be made that the cross coupling between $E_{0,x}$ and $E_{0,y}$ is weak and can be neglected. Under this condition, the governing equation separates into two independent equations, one for each LP mode.

$$\begin{bmatrix} \Omega_{xx} & \Omega_{xy} \\ \Omega_{yx} & \Omega_{yy} \end{bmatrix} \begin{bmatrix} E_{0,x} \\ E_{0,y} \end{bmatrix} - \beta^2 \begin{bmatrix} E_{0,x} \\ E_{0,y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Omega_{xx} E_{0,x} - \beta^2 E_{0,x} = 0$$
$$\Omega_{yy} E_{0,y} - \beta^2 E_{0,y} = 0$$

$$\Omega_{xx} = \frac{1}{\omega}\left(\frac{\partial}{\partial x}\frac{1}{\varepsilon}\frac{\partial}{\partial x} + \omega^2\mu\right)\left(\frac{\partial}{\partial y}\frac{1}{\mu}\frac{\partial}{\partial y} + \omega^2\varepsilon\right) - \frac{1}{\omega^2}\left(\frac{\partial}{\partial x}\frac{1}{\varepsilon}\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x}\frac{1}{\mu}\frac{\partial}{\partial y}\right)$$

$$\Omega_{yy} = \frac{1}{\omega}\left(\frac{\partial}{\partial y}\frac{1}{\varepsilon}\frac{\partial}{\partial y} + \omega^2\mu\right)\left(\frac{\partial}{\partial x}\frac{1}{\mu}\frac{\partial}{\partial x} + \omega^2\varepsilon\right) - \frac{1}{\omega^2}\left(\frac{\partial}{\partial y}\frac{1}{\varepsilon}\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial y}\frac{1}{\mu}\frac{\partial}{\partial x}\right)$$