Electromagnetics:

Electromagnetic Field Theory

Governing Equations for Waveguides

1

Steps for Waveguide Analysis

1. Draw the waveguide.
2. Assume a form of the solution. Outer regions must decay exponentially or be equal to zero.
3. Substitute solution into Maxwell's equations.
4. Simplify equations based on the geometry of the waveguide.
5. Manipulate equations into a differential equation to solve. This is called the *governing equation*.
6. Solve the governing equation in each homogeneous region of the waveguide.
7. "Connect" the solutions in each region using boundary conditions.
8. Calculate the overall field solution.
9. Use the field solution to calculate the waveguide parameters such as β, Z_0, and the profile of the fields.

2

Various Wave Equations

1. Maxwell's Curl Equations

$$\nabla \times \vec{E} = -j\omega\mu\vec{H}$$
$$\nabla \times \vec{H} = j\omega\varepsilon\vec{E}$$

2. Wave Equation in General Media

$$\nabla \times \vec{E} = -j\omega\mu\vec{H} \quad \rightarrow \quad \vec{H} = -\frac{\nabla \times \vec{E}}{j\omega\mu}$$
$$\nabla \times \vec{H} = j\omega\varepsilon\vec{E}$$
$$\nabla \times \left(-\frac{\nabla \times \vec{E}}{j\omega\mu}\right) = j\omega\varepsilon\vec{E}$$
$$\nabla \times \left(\mu^{-1}\nabla \times \vec{E}\right) = \omega^2\varepsilon\vec{E}$$

3. Wave Equation in LHI Media

$$\nabla \times \left(\mu^{-1}\nabla \times \vec{E}\right) = \omega^2\varepsilon\vec{E}$$
$$\nabla \times \nabla \times \vec{E} = \omega^2\mu\varepsilon\vec{E}$$
$$\nabla\left(\nabla \bullet \vec{E}\right) - \nabla^2\vec{E} = k^2\vec{E}$$
$$\nabla^2\vec{E} + k^2\vec{E} = 0$$
$$k = \omega\sqrt{\mu\varepsilon} \equiv \text{wave number}$$

4. Wave Equation Decouples

$$\nabla^2 E_x + k^2 E_x = 0$$
$$\nabla^2 E_y + k^2 E_y = 0$$
$$\nabla^2 E_z + k^2 E_z = 0$$

These equations are solved independently.

EMPossible

Slide 3

3

Expand Maxwell's Equations

Maxwell's equations are used to analyze waveguides.

$$\nabla \times \vec{E} = -j\omega\mu\vec{H} \qquad \nabla \times \vec{H} = j\omega\varepsilon\vec{E}$$

The two curl equations expand into a set of six coupled partial differential equations.

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -j\omega\mu H_x \qquad \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = j\omega\varepsilon E_x$$
$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -j\omega\mu H_y \qquad \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = j\omega\varepsilon E_y$$
$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z \qquad \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\varepsilon E_z$$

There are six field components to solve for: E_x, E_y, E_z, H_x, H_y, and H_z.

Yikes!! ☹

EMPossible

Slide 4

4

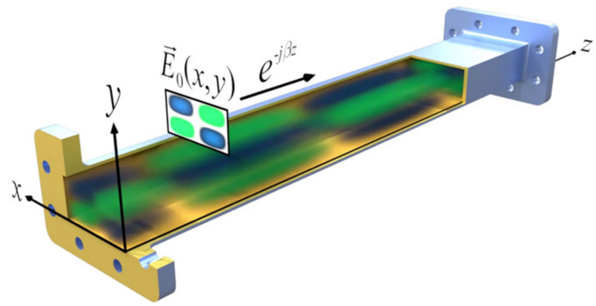
General Form of Solution for Waveguides

A mode in a waveguide has the following general mathematical form.

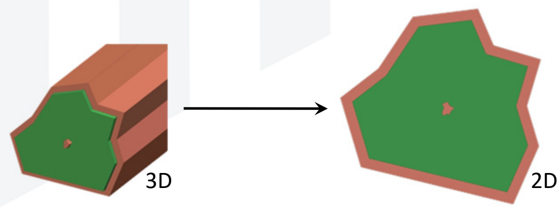
$$\vec{E}(x,y,z) = \vec{E}_0(x,y)\,e^{-j\beta z}$$

complex amplitude, mode shape

accumulation of phase in z direction



This means the problem can be solved by just analyzing the cross section in the xy plane. The problem reduces mathematically to two dimensions.



EMPossible

Slide 5

5

Assume the Form of the Solution

For a waveguide uniform in the z direction, the solution will have the form

$$\vec{E}(x,y,z) = \vec{E}_0(x,y)\,e^{-j\beta z} \qquad \vec{H}(x,y,z) = \vec{H}_0(x,y)\,e^{-j\beta z}$$

Substituting this solution into the set of six equations gives

$$\frac{\partial E_{0,z}}{\partial y} + j\beta E_{0,y} = -j\omega\mu H_{0,x} \qquad \frac{\partial H_{0,z}}{\partial y} + j\beta H_{0,y} = j\omega\varepsilon E_{0,x}$$

$$-j\beta E_{0,x} - \frac{\partial E_{0,z}}{\partial x} = -j\omega\mu H_{0,y} \qquad -j\beta H_{0,x} - \frac{\partial H_{0,z}}{\partial x} = j\omega\varepsilon E_{0,y}$$

$$\frac{\partial E_{0,y}}{\partial x} - \frac{\partial E_{0,x}}{\partial y} = -j\omega\mu H_{0,z} \qquad \frac{\partial H_{0,y}}{\partial x} - \frac{\partial H_{0,x}}{\partial y} = j\omega\varepsilon E_{0,z}$$

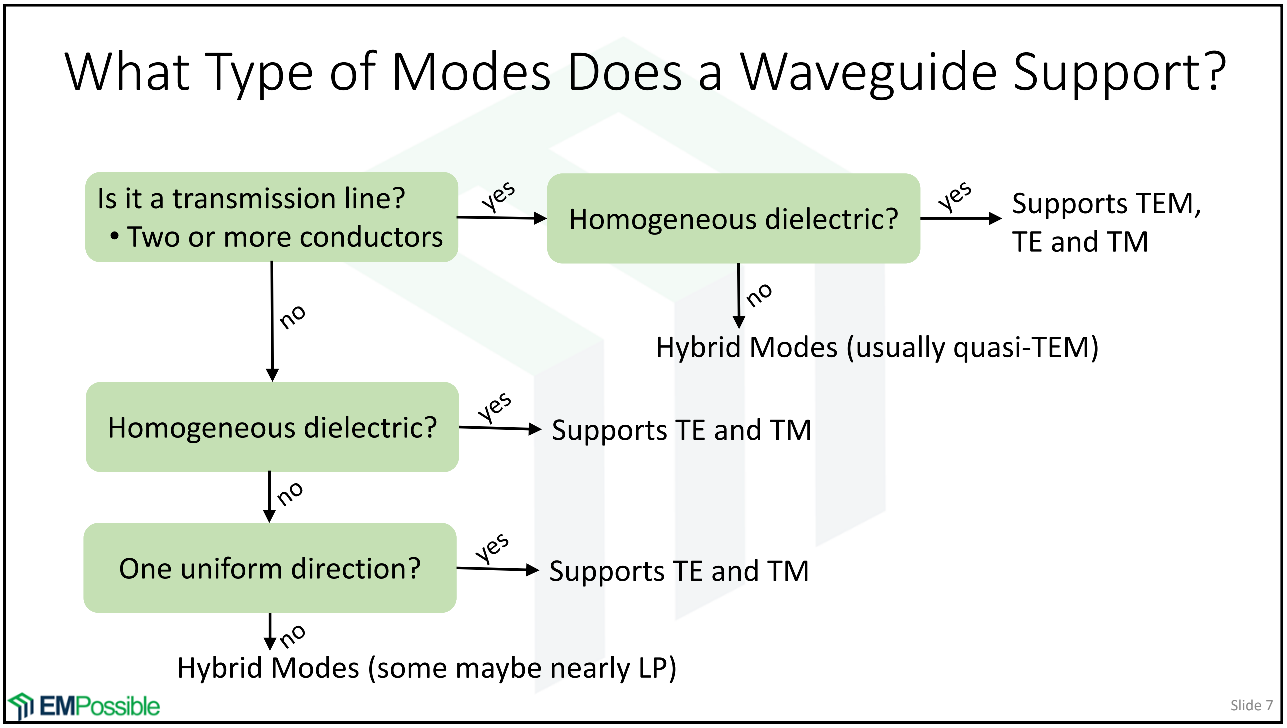
All waveguide analysis starts with these equations.

Things are a little simpler now, but there are still six field components to solve for. ☹

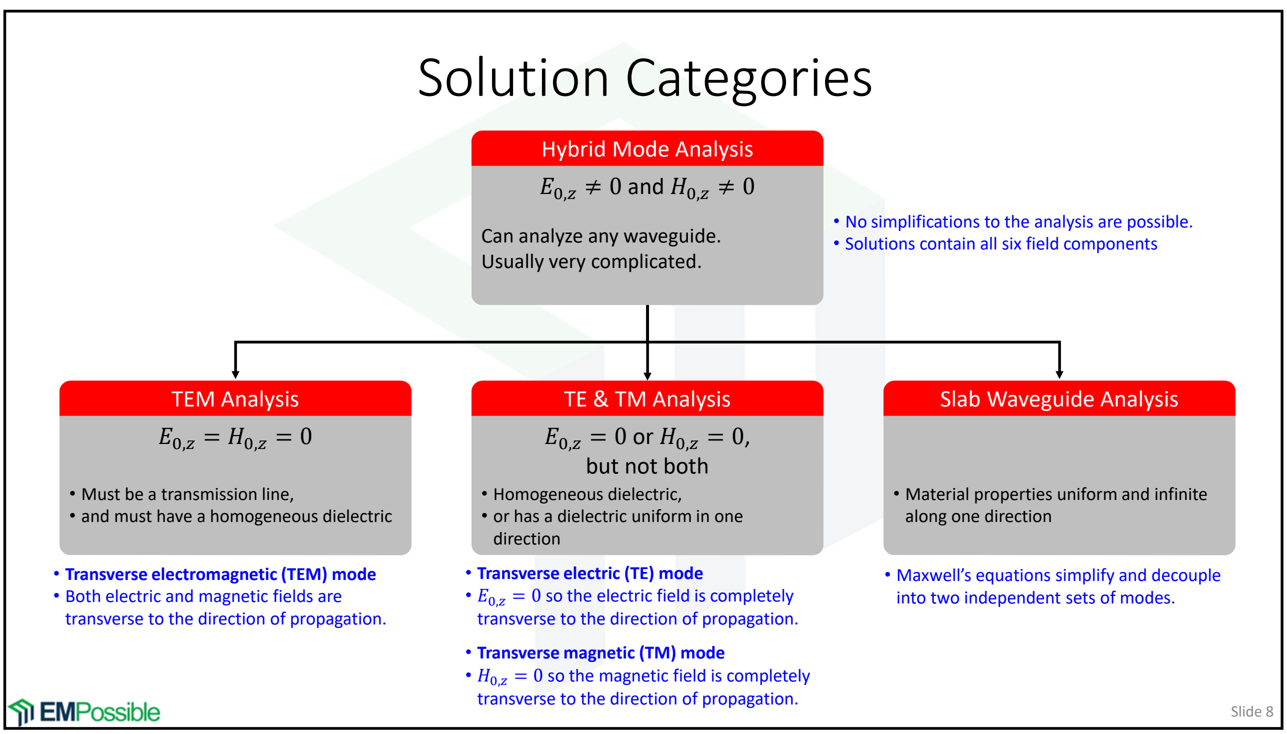
EMPossible

Slide 6

6


What Type of Modes Does a Waveguide Support?
Is it a transmission line?
• Two or more conductors
yes
Homogeneous dielectric?
yes
Supports TEM, TE and TM
no
Hybrid Modes (usually quasi-TEM)
no
Homogeneous dielectric?
yes
Supports TE and TM
no
One uniform direction?
yes
Supports TE and TM
no
Hybrid Modes (some maybe nearly LP)
EMPossible
Slide 7

7


Solution Categories
Hybrid Mode Analysis
$E_{0,z} \neq 0$ and $H_{0,z} \neq 0$
Can analyze any waveguide.
Usually very complicated.
• No simplifications to the analysis are possible.
• Solutions contain all six field components
TEM Analysis
$E_{0,z} = H_{0,z} = 0$
• Must be a transmission line,
• and must have a homogeneous dielectric
• Transverse electromagnetic (TEM) mode
• Both electric and magnetic fields are transverse to the direction of propagation.
TE & TM Analysis
$E_{0,z} = 0$ or $H_{0,z} = 0$, but not both
• Homogeneous dielectric,
• or has a dielectric uniform in one direction
• Transverse electric (TE) mode
• $E_{0,z} = 0$ so the electric field is completely transverse to the direction of propagation.
• Transverse magnetic (TM) mode
• $H_{0,z} = 0$ so the magnetic field is completely transverse to the direction of propagation.
Slab Waveguide Analysis
• Material properties uniform and infinite along one direction
• Maxwell's equations simplify and decouple into two independent sets of modes.
EMPossible
Slide 8

8

Sneak Peek at Equations for Waveguide Analysis

- Identify what types of modes are supported
  - TEM, TE, TM, or hybrid
- Analysis setup

TEM

$$\nabla^2 V = 0$$

TE

$$\nabla^2 H_{0,z} + k_c^2 H_{0,z} = 0$$

TM

$$\nabla^2 E_{0,z} + k_c^2 E_{0,z} = 0$$

Hybrid

$$\begin{bmatrix} -\frac{1}{\omega}\frac{\partial}{\partial x}\frac{1}{\varepsilon}\frac{\partial}{\partial y} & \frac{1}{\omega}\frac{\partial}{\partial x}\frac{1}{\varepsilon}\frac{\partial}{\partial x}+\omega\mu \\ -\left(\frac{1}{\omega}\frac{\partial}{\partial y}\frac{1}{\varepsilon}\frac{\partial}{\partial y}+\omega\mu\right) & \frac{1}{\omega}\frac{\partial}{\partial y}\frac{1}{\varepsilon}\frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} \frac{1}{\omega}\frac{\partial}{\partial x}\frac{1}{\mu}\frac{\partial}{\partial y} & -\left(\frac{1}{\omega}\frac{\partial}{\partial x}\frac{1}{\mu}\frac{\partial}{\partial x}+\omega\varepsilon\right) \\ \frac{1}{\omega}\frac{\partial}{\partial y}\frac{1}{\mu}\frac{\partial}{\partial y}+\omega\varepsilon & -\frac{1}{\omega}\frac{\partial}{\partial y}\frac{1}{\mu}\frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} E_{0,x} \\ E_{0,y} \end{bmatrix} - \beta^2 \begin{bmatrix} E_{0,x} \\ E_{0,y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Slab

$$\text{TE: } \mu\frac{\partial}{\partial x}\left(\frac{1}{\mu}\frac{\partial E_{0,y}}{\partial x}\right) + k_c^2 E_{0,y} = 0$$

$$\text{TM: } \varepsilon\frac{\partial}{\partial x}\left(\frac{1}{\varepsilon}\frac{\partial H_{0,y}}{\partial x}\right) + k_c^2 H_{0,y} = 0$$

EMPossible

Slide 9

9

10