Electromagnetics:
Electromagnetic Field Theory

TE Analysis of Parallel Plate Waveguide

1

Lecture Outline

- TE Analysis
- Analysis of TE Solution
- Visualization of TE Modes
- Conclusions

Slide 2

2

TE Analysis

Slide 3

3

Recall the Starting Point

The governing equation for TE analysis (i.e. $E_{0,z} = 0$) is

$$\frac{\partial^2 H_{0,z}}{\partial x^2} + \frac{\partial^2 H_{0,z}}{\partial y^2} - k_c^2 H_{0,z} = 0 \qquad k_c^2 = k^2 - \beta^2$$

After a solution is obtained, the remaining field components are calculated according to

$$H_{0,x} = -\frac{j\beta}{k_c^2}\frac{\partial H_{0,z}}{\partial x} \qquad E_{0,x} = -\frac{j\omega\mu}{k_c^2}\frac{\partial H_{0,z}}{\partial y}$$

$$H_{0,y} = -\frac{j\beta}{k_c^2}\frac{\partial H_{0,z}}{\partial y} \qquad E_{0,y} = \frac{j\omega\mu}{k_c^2}\frac{\partial H_{0,z}}{\partial x}$$

$$E_{0,z} = 0$$

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Slide 4

4

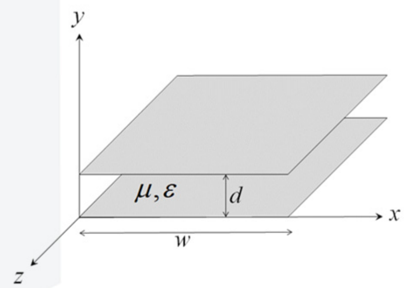
Simplify Governing Equation

Assuming the waveguide is uniform in the direction of x

$$\frac{\partial}{\partial x}=0 \quad \text{and} \quad \frac{\partial^2}{\partial x^2}=0$$

The governing equation reduces to

$$\cancel{\frac{\partial^2 H_{0,z}}{\partial x^2}}+\frac{\partial^2 H_{0,z}}{\partial y^2}-k_c^2 H_{0,z}=0 \quad \rightarrow \quad \frac{d^2 H_{0,z}}{dy^2}-k_c^2 H_{0,z}=0$$



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5

General Solution

The general solution to the governing equation is

$$\frac{d^2 H_{0,z}}{dy^2}-k_c^2 H_{0,z}=0 \quad \rightarrow \quad H_{0,z}=A\sin(k_c y)+B\cos(k_c y)$$

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Slide 6

6

Boundary Conditions (1 of 2)

The tangential component of the electric field $\vec{E}$ must be zero at the plates.

The solution, however, is in terms of the magnetic field $H_{0,z}$.

The solution must be written in terms of an electric field that is tangential to the plates.

The only component of the electric field tangential to the plates is $E_{0,x}$.

$$E_{0,x} = -\frac{j\omega\mu}{k_c^2}\frac{\partial H_{0,z}}{\partial y} = -\frac{j\omega\mu}{k_c^2}\frac{\partial}{\partial y}\left[A\sin(k_c y) + B\cos(k_c y)\right]$$

$$= -\frac{j\omega\mu}{k_c}\left[A\cos(k_c y) - B\sin(k_c y)\right]$$

Boundary Conditions (2 of 2)

The first boundary condition is

$$E_{0,x}(x,0) = 0 \qquad E_{0,x}(x,0) = -\frac{j\omega\mu}{k_c}\left[A\cos(0) - B\sin(0)\right] = -\frac{j\omega\mu}{k_c}A = 0 \quad \rightarrow \quad A = 0$$

The second boundary condition is

$$E_{0,x}(x,d) = 0 \qquad E_{0,x}(x,d) = -\frac{j\omega\mu}{k_c}\left[-B\sin(k_c d)\right] = B\frac{j\omega\mu}{k_c}\sin(k_c d)$$

$B = 0$ cannot be chosen because that would lead to a trivial solution. Instead, it must be the $\sin(k_c d)$ term that is zero at $y = d$.

$$\sin(k_c d) = 0 \quad \rightarrow \quad k_c d = m\pi \quad m = 1, 2, 3, \ldots$$

Note that $m = 0$ would force the entire field to be zero so this is not a valid solution. Not obvious yet.

The cutoff wave number k_c is then

$$k_c = \frac{m\pi}{d} \quad m = 1, 2, 3, \ldots$$

Remember this equation k_c for the next slide.

Phase Constant β (1 of 2)

Recall the original definition of the cutoff wave number k_c. Solve this equation for β.

$$k_c^2 = k^2 - \beta^2 \quad \rightarrow \quad \beta = \sqrt{k^2 - k_c^2}$$

An expression for k_c was derived on the previous slide that arose from the boundary conditions.

$$\beta = \sqrt{k^2 - \left(\frac{m\pi}{d}\right)^2} \qquad m = 1, 2, 3, \ldots$$

Since m is only integer values, it is concluded that there are an infinite number of discrete solutions and the order of the mode is m.

$$\boxed{\beta_m = \sqrt{k^2 - \left(\frac{m\pi}{d}\right)^2} \qquad m = 1, 2, 3, \ldots}$$

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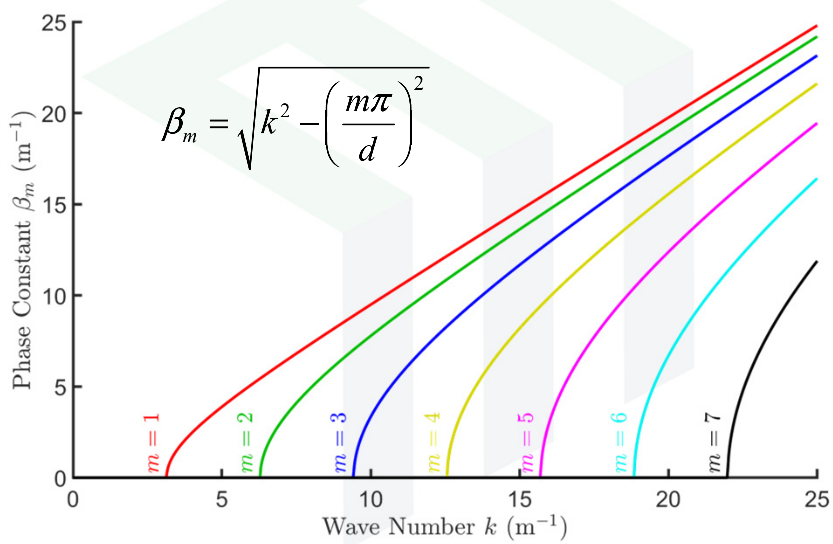
Slide 9

9

Phase Constant β (2 of 2)



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Slide 10

10

Final Solution

Recall the general solution was

$$\frac{d^2 H_{0,z}}{dy^2} - k_c^2 H_{0,z} = 0 \quad \rightarrow \quad H_{0,z} = A\sin(k_c y) + B\cos(k_c y)$$

But now it is known that $A = 0$ and $k_c = m\pi/d$. The final solution is

$$H_{0,z}(x,y) = B_m \cos\left(\frac{m\pi y}{d}\right) \quad \rightarrow \quad \boxed{H_z(x,y,z) = B_m \cos\left(\frac{m\pi y}{d}\right) e^{-j\beta_m z}}$$

The remaining field components are calculated from this result.

$$H_x(x,y,z) = -\frac{j\beta_m}{k_c^2}\frac{\partial H_z}{\partial x} = -\frac{j\beta_m}{k_c^2}\frac{\partial}{\partial x}\left[B_m \cos\left(\frac{m\pi y}{d}\right)e^{-j\beta_m z}\right] = 0$$

$$H_y(x,y,z) = -\frac{j\beta_m}{k_c^2}\frac{\partial H_z}{\partial y} = -\frac{j\beta_m}{k_c^2}\frac{\partial}{\partial y}\left[B_m \cos\left(\frac{m\pi y}{d}\right)e^{-j\beta_m z}\right] = \frac{j\beta_m}{k_c} B_m \sin\left(\frac{m\pi y}{d}\right)e^{-j\beta_m z}$$

$$E_x(x,y,z) = -\frac{j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial y} = -\frac{j\omega\mu}{k_c^2}\frac{\partial}{\partial y}\left[B_m \cos\left(\frac{m\pi y}{d}\right)e^{-j\beta_m z}\right] = \frac{j\omega\mu}{k_c} B_m \sin\left(\frac{m\pi y}{d}\right)e^{-j\beta_m z}$$

$$E_y(x,y,z) = \frac{j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial x} = \frac{j\omega\mu}{k_c^2}\frac{\partial}{\partial x}\left[B_m \cos\left(\frac{m\pi y}{d}\right)e^{-j\beta_n z}\right] = 0$$

$$E_z(x,y,z) = 0$$

Note that $m = 0$ would force the entire field to be zero so this is not a valid solution. Now it is obvious.

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Slide 11

11

Analysis of TE Solution

Slide 12

12

Why No TE_0 Mode?

For $m = 0$, the field components would be

$$H_x = 0$$
$$H_y = \frac{j\beta_m}{k_c} B_m \sin(0) e^{-j\beta_m z} = 0$$
$$H_z = B_m \cos(0) e^{-j\beta_m z} = B_m e^{-j\beta_m z}$$
$$E_x = \frac{j\omega\mu}{k_c} B_m \sin(0) e^{-j\beta_m z} = 0$$
$$E_y = 0$$
$$E_z = 0$$

This is not a physical solution because the electric field is entirely zero. This would not satisfy Maxwell's equations.

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Slide 13

13

Cutoff Condition

Recall the phase constant β_m is calculated as

$$\beta_m = \sqrt{k^2 - k_c^2} = \sqrt{k^2 - \left(\frac{m\pi}{d}\right)^2} \qquad m = 1, 2, 3, \ldots$$

This becomes imaginary when $k_c > k$. Values of m that cause this condition correspond to modes that are "cutoff." These are still modes, but they decay very quickly so they are almost never used and are not considered *guided* modes.

$$k_c = \omega_c \sqrt{\mu\varepsilon} = \frac{m\pi}{d}$$

$$2\pi f_c \sqrt{\mu\varepsilon} = \frac{m\pi}{d}$$

$$f_c = \frac{m}{2d\sqrt{\mu\varepsilon}} = \frac{k_c}{2\pi\sqrt{\mu\varepsilon}}$$

→ This is the cutoff frequency for the TE_m mode.

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14

Characteristic Impedance Z_{TE}

The characteristic impedance of the TE mode is defined as

$$Z_{\mathrm{TE}} = \frac{E_{0,x}}{H_{0,y}} = -\frac{E_{0,y}}{H_{0,x}}$$

It is derived by substituting in the expressions for the field components.

$$Z_{\mathrm{TE}} = \frac{E_{0,x}}{H_{0,y}} = \frac{\dfrac{j\omega\mu}{k_{\mathrm{c}}} B_m \sin\left(\dfrac{n\pi y}{d}\right)}{\dfrac{j\beta_m}{k_{\mathrm{c}}} B_m \sin\left(\dfrac{n\pi y}{d}\right)} = \frac{\omega\mu}{\beta_m} = \eta\frac{k}{\beta_m}$$

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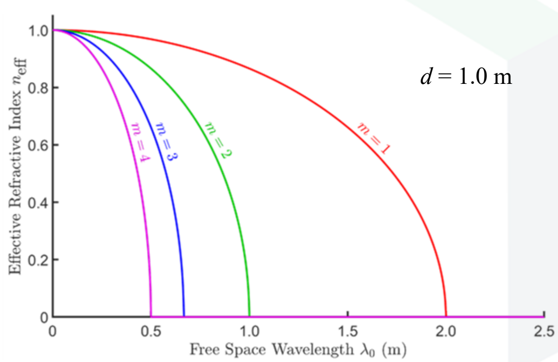
15

Effective Refractive Index n_{eff}

A wave propagates in a waveguide at a speed quantified by the effective refractive index n_{eff}.

$$\beta_m = k_0 n_{\mathrm{eff}} = \sqrt{k^2 - \left(\frac{m\pi}{d}\right)^2} \quad \rightarrow \quad n_{\mathrm{eff}} = n\sqrt{1 - \left(\frac{m\lambda_0}{2nd}\right)^2}$$



This term acts to make $n_{\mathrm{eff}} < n$.

n is the refractive index of the dielectric in the waveguide.

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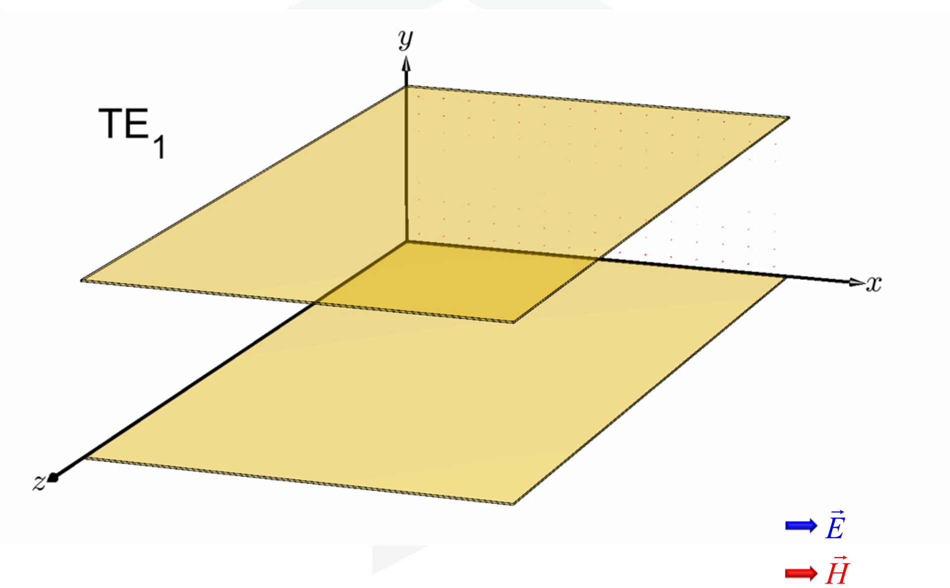
Slide 16

16

Visualization of TE Modes

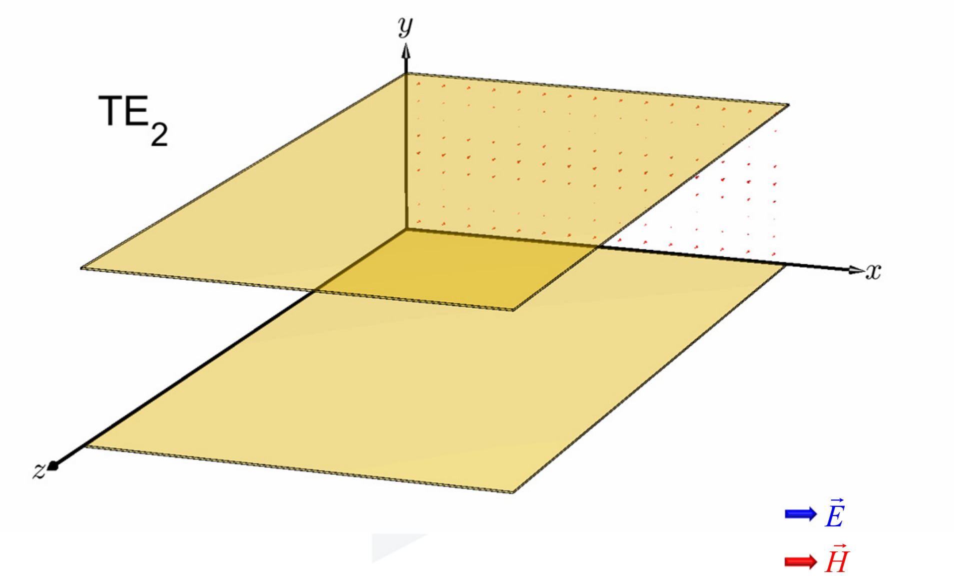
17

Visualization of TE_1 Mode



18

Visualization of TE_2 Mode



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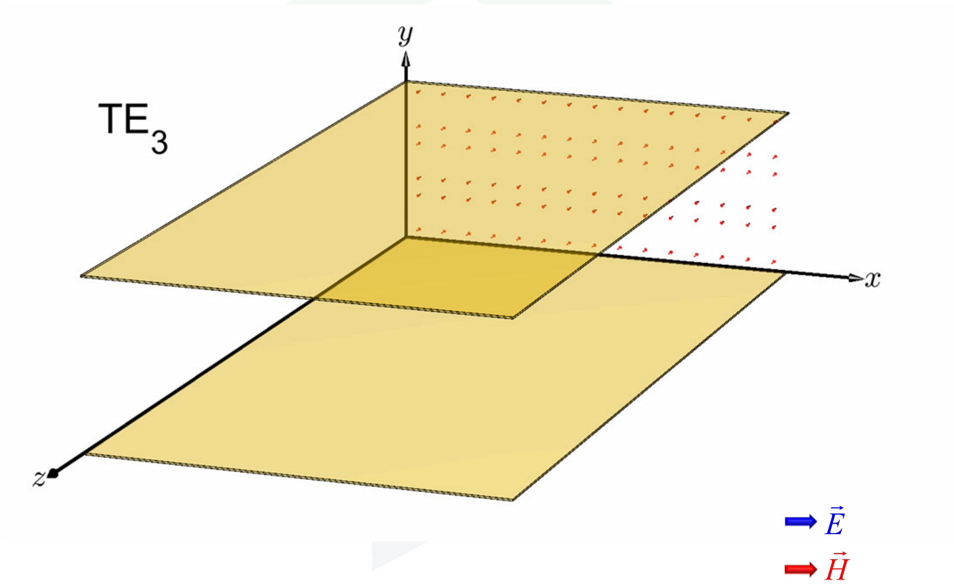
19

Visualization of TE_3 Mode



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20

Conclusion

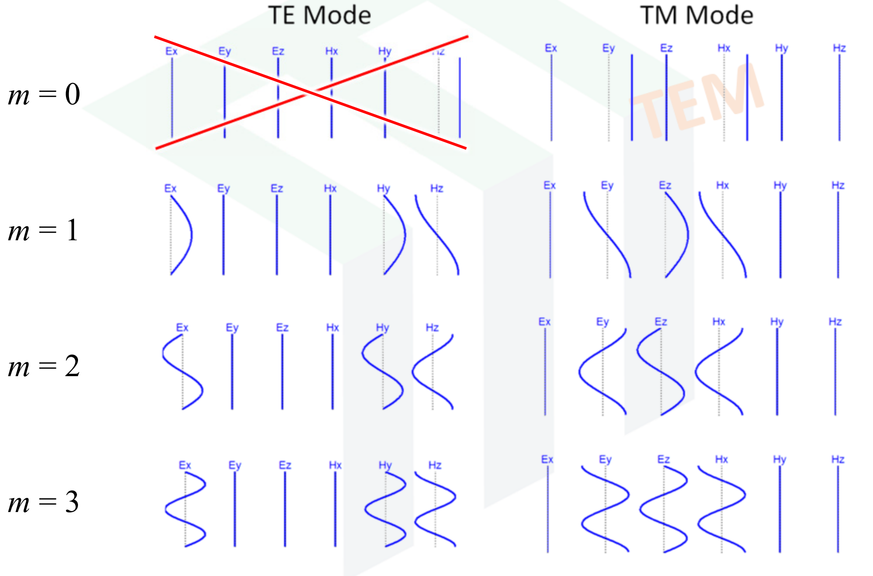
21

Summary of Parallel Plate Waveguide

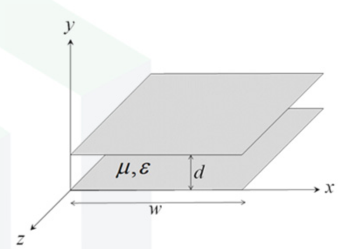
Parameter	TEM	TM$_m$ $m = 0,1,2,3\ldots$	TE$_m$ $m = 1,2,3\ldots$
k	$\omega\sqrt{\mu\varepsilon}$	$\omega\sqrt{\mu\varepsilon}$	$\omega\sqrt{\mu\varepsilon}$
k_c	0	$m\pi / d$	$m\pi / d$
β	$k = \omega\sqrt{\mu\varepsilon}$	$\sqrt{k^2 - k_c^2}$	$\sqrt{k^2 - k_c^2}$
λ_c	∞	$2\pi/k_c = 2d/n$	$2\pi/k_c = 2d/n$
λ_g	$2\pi / k$	$2\pi / \beta_m$	$2\pi / \beta_m$
v_p	$\omega/k = 1/\sqrt{\mu\varepsilon}$	ω / β_m	ω / β_m
α_d	$k \tan\delta/2$	$k^2 \tan\delta/2\beta_m$	$k^2 \tan\delta/2\beta_m$
α_c	$R_s/\eta d$	$2kR_s/\beta_m\eta d$	$2k_c^2 R_s/k\beta_m\eta d$
E_x	0	0	$(j\omega\mu/k_c) B_m \sin(m\pi y/d) e^{-j\beta_m z}$
E_y	$(-V_0/d) e^{-j\beta z}$	$(-j\beta_m/k_c) A_m \cos(m\pi y/d) e^{-j\beta_m z}$	0
E_z	0	$A_m \sin(m\pi y/d) e^{-j\beta_m z}$	0
H_x	$(-V_0/\eta d) e^{-j\beta z}$	$(j\omega\varepsilon/k_c) A_m \cos(m\pi y/d) e^{-j\beta_m z}$	0
H_y	0	0	$(j\beta_m/k_c) B_m \sin(m\pi y/d) e^{-j\beta_m z}$
H_z	0	0	$B_m \cos(m\pi y/d) e^{-j\beta_m z}$
Z	$\eta d/w$	$\beta_m\eta/k$	$k\eta/\beta_m$

22

Modes in Parallel Plate Waveguide



Summary of TE Analysis



Field Solution

$$E_x(x,y,z)=\frac{j\omega\mu}{k_c}B_m\sin\left(\frac{m\pi y}{d}\right)e^{-j\beta_m z}$$

$$E_y(x,y,z)=0$$

$$E_z(x,y,z)=0$$

$$H_x(x,y,z)=0$$

$$H_y(x,y,z)=\frac{j\beta_m}{k_c}B_m\sin\left(\frac{m\pi y}{d}\right)e^{-j\beta_m z}$$

$$H_z(x,y,z)=B_m\cos\left(\frac{m\pi y}{d}\right)e^{-j\beta_m z}$$

- TE_0 mode does not exist
- TE_1 is the lowest order TE mode

Phase Constant

$$\beta_m=\sqrt{k^2-\left(\frac{m\pi}{d}\right)^2}$$

$m=1,2,3,\ldots$

Same equation as for TM

Cutoff Frequency

$$f_{c,m}=\frac{m}{2d\sqrt{\mu\varepsilon}}=\frac{mc_0}{2nd}$$

Same equation as for TM

Characteristic Impedance

$$Z_{TE,m}=\frac{k\eta}{\beta_m}$$

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