Electromagnetics:
Electromagnetic Field Theory

TEM Analysis of the Parallel Plate Waveguide

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Lecture Outline

- TEM Analysis
- Example

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TEM Analysis

Starting Point for TEM Analysis

Assuming the parallel plate waveguide has an LHI dielectric between the plates, start with the homogeneous Laplaces' equation.

$$\nabla^2 V(x,y,z) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

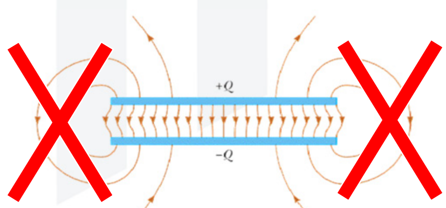
The parallel plate waveguide is uniform in the x and z directions so the governing equation $\nabla^2 V = 0$ reduces to

$$\cancel{\frac{\partial^2 V}{\partial x^2}} + \frac{\partial^2 V}{\partial y^2} + \cancel{\frac{\partial^2 V}{\partial z^2}} = 0$$

$$\frac{\partial^2 V}{\partial y^2} = 0$$

$$\frac{d^2 V}{dy^2} = 0$$

Note, by assuming the field is uniform in the x direction, the fringing fields at the edges are ignored.



(a)

The derivative becomes ordinary because y is the only independent variable left.

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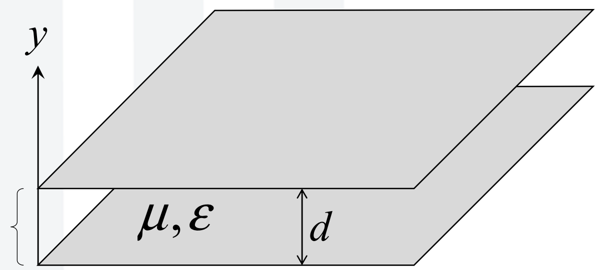
How to Interpret the Governing Equation

The governing equation is now

$$\frac{d^2V}{dy^2} = 0$$

The solution to this will give $V(y)$.

$$V(y) \qquad 0 \le y \le d$$



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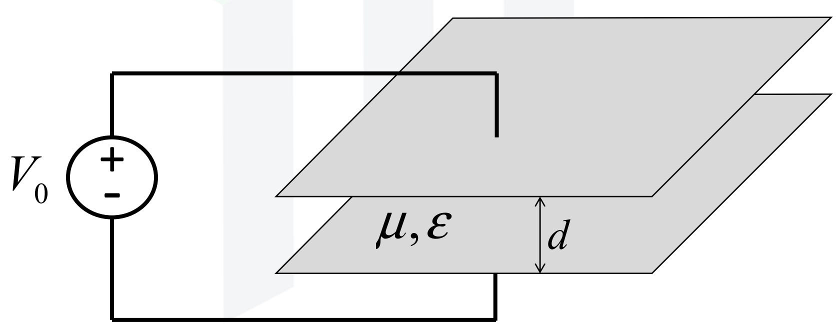
Boundary Conditions

Boundary conditions are needed to solve the differential equation.

$$V(0) = ?$$
$$V(d) = ?$$

Apply a voltage V_0 across the plates and the boundary conditions will be

$$V(0) = 0$$
$$V(d) = V_0$$



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General Solution to Differential Equation

The differential equation with boundary conditions is

$$\frac{d^2V}{dy^2} = 0 \qquad 0 \le y \le d \qquad\qquad V(0) = 0 \ \text{ and } \ V(d) = V_0$$

This is solved by integrating with respect to y twice.

$$\frac{d^2V}{dy^2} = 0$$

$$\frac{dV}{dy} = A$$

$$V(y) = Ay + B$$

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Apply Boundary Conditions

The general solution is now

$$V(y) = Ay + B$$

Apply the boundary condition at $y = 0$.

$$V(0) = 0$$

$$A \cdot 0 + B = 0$$

$$B = 0$$

Apply the boundary condition at $y = d$.

$$V(d) = V_0$$

$$A \cdot d + \cancel{B} = V_0$$

$$A \cdot d = V_0$$

$$A = V_0 / d$$

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The Solution (1 of 2)

The final solution to the governing equation is

$$V(y) = \frac{V_0}{d} y$$

The analysis is still not finished because nothing was learned about the waveguide.

The electric field is calculated from the electric potential to be

$$\vec{E} = -\nabla V = -\hat{a}_x \frac{\partial V}{\partial x} - \hat{a}_y \frac{\partial V}{\partial y} - \hat{a}_z \frac{\partial V}{\partial z}$$

$$\vec{E} = -\hat{a}_x \frac{\partial}{\partial x}\left(\frac{V_0}{d} y\right) - \hat{a}_y \frac{\partial}{\partial y}\left(\frac{V_0}{d} y\right) - \hat{a}_z \frac{\partial}{\partial z}\left(\frac{V_0}{d} y\right)$$

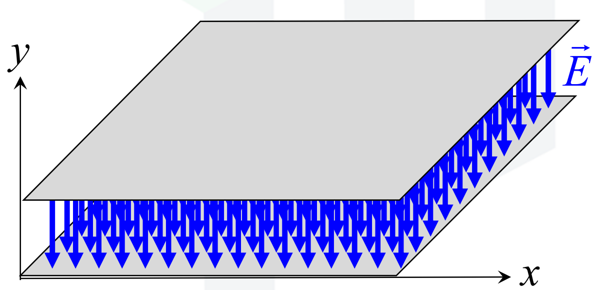
$$\vec{E} = -\hat{a}_x \cancel{\frac{\partial}{\partial x}\left(\frac{V_0}{d} y\right)} - \hat{a}_y \frac{V_0}{d} - \hat{a}_z \cancel{\frac{\partial}{\partial z}\left(\frac{V_0}{d} y\right)}$$

$$\vec{E} = -\hat{a}_y \frac{V_0}{d}$$

The Solution (2 of 2)

If the fringing fields are ignored outside of the waveguide, the electric field is expressed as

$$\vec{E}(x,y) = \begin{cases} -\hat{a}_y \dfrac{V_0}{d} & \text{for } 0 \le x \le w \text{ and } 0 \le y \le d \\ 0 & \text{otherwise} \end{cases}$$



The Wave Solution

The TEM wave was derived by way of an electrostatic analysis. This ignores the wave propagating nature of a TEM wave. To account for propagation in the z direction, $e^{-j\beta z}$ must be incorporated to account for accumulation of phase in the z direction.

$$\vec{E}(x,y,z)=\begin{cases}-\hat{a}_y \dfrac{V_0}{d}e^{-j\beta z} & \text{for } 0\le x\le w \text{ and } 0\le y\le d\\ 0 & \text{otherwise}\end{cases}$$

It follows that the magnetic field component is

$$\vec{H}(x,y,z)=\frac{\hat{a}_z\times\vec{E}}{\eta}=\frac{\hat{a}_z\times\left(-\hat{a}_y\dfrac{V_0}{d}e^{-j\beta z}\right)}{\eta}=-\left(\hat{a}_z\times\hat{a}_y\right)\frac{V_0}{\eta d}e^{-j\beta z}=\hat{a}_x\frac{V_0}{\eta d}e^{-j\beta z}$$

$$\vec{H}(x,y,z)=\begin{cases}\hat{a}_x \dfrac{V_0}{\eta d}e^{-j\beta z} & \text{for } 0\le x\le w \text{ and } 0\le y\le d\\ 0 & \text{otherwise}\end{cases}$$

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Impedance from Wave Solution (1 of 2)

The impedance Z_{TEM} of the TEM wave is defined as

$$Z_{\text{TEM}}=\frac{V_0}{I}$$

The current term I must be determined. Recall the magnetic field $\vec{H}$ above an infinite current sheet is

$$\vec{H}_{1\text{ sheet}}=\frac{\vec{K}\times\hat{n}}{2}$$

$\vec{K}\equiv$ surface current density (A/m) $\hat{n}=-\hat{a}_y$

Using this equation for the parallel plate waveguide ignores fringing fields at the edges.

It follows that the field between two current sheets (i.e. in the parallel plate waveguide) is

$$\vec{H}_{2\text{ sheets}}=2\vec{H}_{1\text{ sheet}}=\vec{K}\times\hat{n}$$

Solving this for the surface current $\vec{K}$ yields

$$\vec{K}=\hat{n}\times\vec{H}=\left(-\hat{a}_y\right)\times\vec{H}=\vec{H}\times\hat{a}_y$$

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Impedance from Wave Solution (2 of 2)

Find the total current I by integrating the surface current across the plate.

$$I = \int_0^w \left(\vec{K} \bullet \hat{a}_z \right) dx = \int_0^w \left[\left(\vec{H} \times \hat{a}_y \right) \bullet \hat{a}_z \right] dx = \int_0^w H_x dx$$

Let $z = 0$ and the magnetic field $\vec{H}$ reduces to

$$H_x \left(z = 0 \right) = \frac{V_0}{\eta d}$$

Substituting $H_x(z = 0)$ into the equation for I leads to

$$I = \int_0^w \left(\frac{V_0}{\eta d} \right) dx = \frac{V_0}{\eta d} \int_0^w dx = \frac{V_0}{\eta d} w = \frac{w}{d} \frac{V_0}{\eta}$$

The characteristic impedance Z_{TEM} is found by substituting this into the original definition.

$$Z_{\text{TEM}} = \frac{V_0}{I} = \frac{V_0}{\frac{w}{d} \frac{V_0}{\eta}} = \eta \frac{d}{w}$$

$$\boxed{Z_{\text{TEM}} = \eta \frac{d}{w}}$$

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Propagation Constant β_{TEM}

The phase constant β_{TEM} cannot be calculated from the present solution because it was analyzed using an electrostatic approximation where there is no propagation.

Recall for TEM modes that $\beta = k$. This implies that TEM waves propagate at the same speed as a plane wave in an infinite medium composed of the dielectric that resides between the plates.

$$\boxed{\beta_{\text{TEM}} \cong \omega \sqrt{\mu \varepsilon}}$$

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Distributed Inductance and Capacitance

The characteristic impedance Z_{TEM} of the parallel plate transmission line is

$$Z_{\text{TEM}} = \eta \frac{d}{w}$$

The distributed capacitance C can be estimated by looking at the transmission like a parallel plate capacitor.

$$C = \varepsilon \frac{w}{d}$$

It follows that the distributed inductance L is

$$Z_{\text{TEM}} = \eta \frac{d}{w} = \sqrt{\frac{L}{C}} = \sqrt{\frac{L}{\varepsilon (w/d)}} \qquad L = \mu \frac{d}{w}$$

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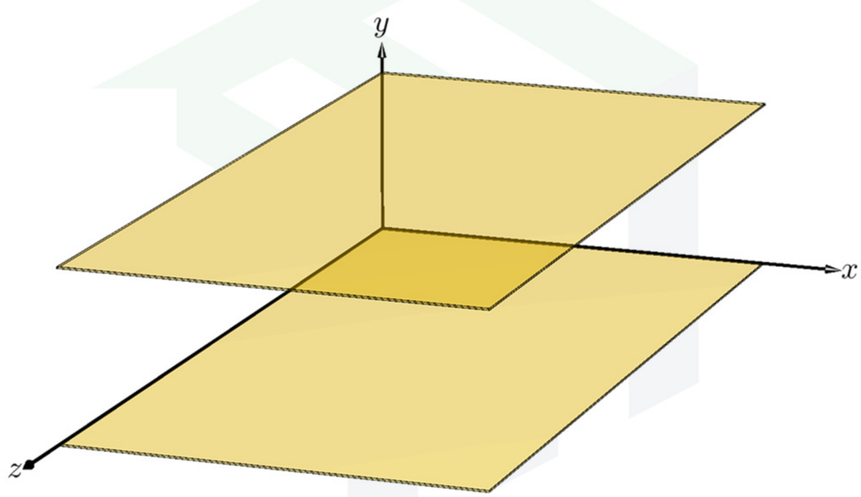
Visualization of TEM Mode



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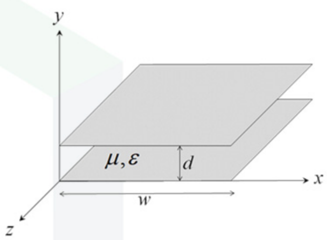
Summary of TEM Analysis



Field Solution

$E_x(x,y,z) = 0$

$E_y(x,y,z) = -\frac{V_0}{d} e^{-j\beta z}$

$E_z(x,y,z) = 0$

$H_x(x,y,z) = \frac{V_0}{\eta d} e^{-j\beta z}$

$H_y(x,y,z) = 0$

$H_z(x,y,z) = 0$

- TEM has no cutoff frequency
- TEM is the TM_0 mode.

Phase Constant

$\beta_{\text{TEM}} \cong \omega \sqrt{\mu \varepsilon}$

Same as plane wave.

Cutoff Frequency

$f_c = 0$

No cutoff frequency.
Mode supported at DC.

Characteristic Impedance

$Z_{\text{TEM}} = \eta \frac{d}{w}$

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Example

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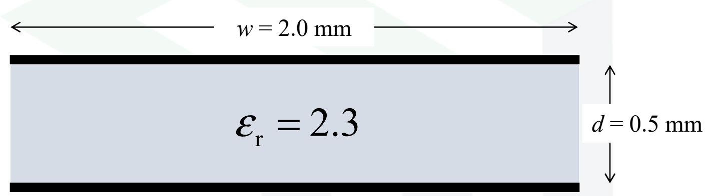
Example #1 (1 of 3)

Given the following parallel plate waveguide...



What is the characteristic impedance Z_{TEM}?

What value of w would make this transmission line 50 Ω?

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Example #1 (2 of 3)

The equation for characteristic impedance Z_{TEM} is

$$Z_{\text{TEM}} = \eta \frac{d}{w}$$

The impedance η of the dielectric is

$$\eta = \eta_0 \sqrt{\frac{\mu_\text{r}}{\varepsilon_\text{r}}} = (376.73\ \Omega)\sqrt{\frac{1.0}{2.3}} = 248.4\ \Omega$$

The characteristic impedance Z_{TEM} is therefore

$$Z_{\text{TEM}} = \frac{\eta d}{w} = \frac{(248.4\ \Omega)(0.5\ \text{mm})}{(2.0\ \text{mm})} = \boxed{62.1\ \Omega}$$

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Example #1 (3 of 3)

Solve the equation for characteristic impedance for w.

$$Z_{\text{TEM}} = \eta \frac{d}{w} \quad \rightarrow \quad w = \frac{\eta d}{Z_{\text{TEM}}}$$

To get 50 Ω, w must be

$$w = \frac{\eta d}{Z_0} = \frac{(248.4\ \Omega)(0.5\ \text{mm})}{(50\ \Omega)} = \boxed{2.48\ \text{mm}}$$

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