Electromagnetics:
Electromagnetic Field Theory

TM Analysis of the Parallel Plate Waveguide

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Lecture Outline

- TM Analysis
- Analysis of TM Solution
- Visualization of the TM Modes
- Example

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TM Analysis

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Recall the Starting Point

The governing equation for TM analysis (i.e. $H_{0,z} = 0$) is

$$\frac{\partial^2 E_{0,z}}{\partial x^2} + \frac{\partial^2 E_{0,z}}{\partial y^2} - k_c^2 E_{0,z} = 0 \qquad k_c^2 = k^2 - \beta^2$$

After a solution is obtained, the remaining field components are calculated according to

$$H_{0,x} = \frac{j\omega\varepsilon}{k_c^2}\frac{\partial E_{0,z}}{\partial y} \qquad E_{0,x} = -\frac{j\beta}{k_c^2}\frac{\partial E_{0,z}}{\partial x}$$

$$H_{0,y} = -\frac{j\omega\varepsilon}{k_c^2}\frac{\partial E_{0,z}}{\partial x} \qquad E_{0,y} = -\frac{j\beta}{k_c^2}\frac{\partial E_{0,z}}{\partial y}$$

$$H_{0,z} = 0$$

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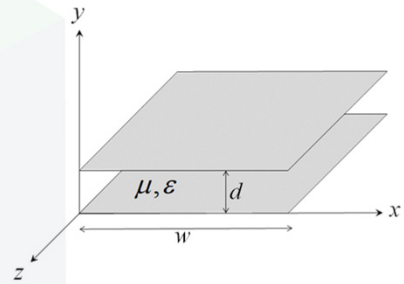
Simplify Governing Equation

Assuming the waveguide is uniform in the direction of x

$$\frac{\partial}{\partial x} = 0 \quad \text{and} \quad \frac{\partial^2}{\partial x^2} = 0$$



The governing equation reduces to

$$\cancel{\frac{\partial^2 E_{0,z}}{\partial x^2}} + \frac{\partial^2 E_{0,z}}{\partial y^2} - k_c^2 E_{0,z} = 0 \quad \rightarrow \quad \frac{d^2 E_{0,z}}{dy^2} - k_c^2 E_{0,z} = 0$$

General Solution

The general solution to the governing equation is

$$\frac{d^2 E_{0,z}}{dy^2} - k_c^2 E_{0,z} = 0 \quad \rightarrow \quad E_{0,z} = A\sin(k_c y) + B\cos(k_c y)$$

Boundary Conditions

The electric field component $E_{0,z}$ is tangential to the interfaces. So, the boundary conditions are applied to this directly.

The first boundary condition is

$$E_{0,z}(x,0) = A\sin(0) + B\cos(0) = B = 0 \quad \rightarrow \quad B = 0$$

The second boundary condition is

$$E_{0,z}(x,d) = A\sin(k_c d) = 0$$

$A = 0$ cannot be chosen because that would lead to a trivial solution. Instead, it must be the $\sin(k_c d)$ term that is zero at $y = d$.

$$\sin(k_c d) = 0 \quad \rightarrow \quad k_c d = m\pi \quad m = 0,1,2,3,\ldots$$

The cutoff wave number is then

$$k_c = \frac{m\pi}{d} \quad m = 0,1,2,3,\ldots$$

Note that $m = 0$ is allowed in this case because it does not force the field to be entirely zero (not obvious yet). It does, however, force the field to be perfectly uniform. Thus, TM_0 is the TEM mode.

Phase Constant β

Recall the original definition of the cutoff wave number. Solve equation this for β.

$$k_c^2 = k^2 - \beta^2 \quad \rightarrow \quad \beta = \sqrt{k^2 - k_c^2}$$

Substitute in $k_c = m\pi/d$ from the previous slide to get

$$\beta = \sqrt{k^2 - \left(\frac{m\pi}{d}\right)^2} \qquad m = 0,1,2,3,\ldots$$

It is observed that there is an infinite number of discrete solutions where the order of the mode is m.

$$\boxed{\beta_m = \sqrt{k^2 - \left(\frac{m\pi}{d}\right)^2} \qquad m = 0,1,2,3,\ldots}$$

TM_0 is the TEM mode

Final Solution

Recall that the general solution was

$$\frac{d^2E_{0,z}}{dy^2} - k_c^2 E_{0,z} = 0 \quad \rightarrow \quad E_{0,z} = A\sin(k_c y) + B\cos(k_c y)$$

But now it is known that $B = 0$ and $k_c = m\pi/d$. The final solution is

$$E_{0,z}(x,y) = A_m \sin\left(\frac{m\pi y}{d}\right) \quad \rightarrow \quad \boxed{E_z(x,y,z) = A_m \sin\left(\frac{m\pi y}{d}\right) e^{-j\beta_m z}}$$

The remaining field components are calculated from this result.

$$H_x(x,y,z) = \frac{j\omega\varepsilon}{k_c^2}\frac{\partial E_z}{\partial y} = \frac{j\omega\varepsilon}{k_c^2}\frac{\partial}{\partial y}\left[A_m \sin\left(\frac{m\pi y}{d}\right)e^{-j\beta_m z}\right] = \frac{j\omega\varepsilon}{k_c} A_m \cos\left(\frac{m\pi y}{d}\right)e^{-j\beta_m z}$$

$$H_y(x,y,z) = -\frac{j\omega\varepsilon}{k_c^2}\frac{\partial E_z}{\partial x} = -\frac{j\omega\varepsilon}{k_c^2}\frac{\partial}{\partial x}\left[A_m \sin\left(\frac{m\pi y}{d}\right)e^{-j\beta_m z}\right] = 0$$

$$H_z(x,y,z) = 0$$

$$E_x(x,y,z) = -\frac{j\beta_m}{k_c^2}\frac{\partial E_z}{\partial x} = -\frac{j\beta_m}{k_c^2}\frac{\partial}{\partial x}\left[A_m \sin\left(\frac{m\pi y}{d}\right)e^{-j\beta_m z}\right] = 0$$

$$E_y(x,y,z) = -\frac{j\beta_m}{k_c^2}\frac{\partial E_z}{\partial y} = -\frac{j\beta_m}{k_c^2}\frac{\partial}{\partial y}\left[A_m \sin\left(\frac{m\pi y}{d}\right)e^{-j\beta_m z}\right] = -\frac{j\beta_m}{k_c} A_m \cos\left(\frac{m\pi y}{d}\right)e^{-j\beta_m z}$$

Note that $m = 0$ is allowed in this case because it does not force the field to be entirely zero (obvious here). It also forces the field to be perfectly uniform. Thus, TM_0 is the TEM mode.

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Analysis of TM Solution

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Why Does TM_0 Mode Exist?

For $m = 0$, the field components are

$$H_x = \frac{j\omega\varepsilon}{k_c} A_m \cos(0) e^{-j\beta_m z} \quad = \frac{j\omega\varepsilon}{k_c} A_m e^{-j\beta_m z}$$
$$H_y = 0$$
$$H_z = 0$$
$$E_x = 0$$
$$E_y = -\frac{j\beta_m}{k_c} A_m \cos(0) e^{-j\beta_m z} \quad = -\frac{j\beta_m}{k_c} A_m e^{-j\beta_m z}$$
$$E_z(x,y,z) = A_m \sin(0) e^{-j\beta_m z} \quad = 0$$

This is a valid solution because it satisfies Maxwell's equations and it is a non-trivial solution.

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Cutoff Condition

Recall that the phase constant β_m is calculated as

$$\beta_m = \sqrt{k^2 - k_c^2} = \sqrt{k^2 - \left(\frac{m\pi}{d}\right)^2} \qquad m = 0,1,2,3,\ldots$$

This becomes imaginary when $k_c > k$. Values of m that cause this condition correspond to modes that are "cutoff." These are still modes, but they decay very quickly so they are almost never used and are not considered *guided modes*.

$$k_c = \omega_c \sqrt{\mu\varepsilon} = \frac{m\pi}{d}$$
$$2\pi f_c \sqrt{\mu\varepsilon} = \frac{m\pi}{d}$$
$$f_c = \frac{m}{2d\sqrt{\mu\varepsilon}} = \frac{k_c}{2\pi\sqrt{\mu\varepsilon}}$$

→ This is the cutoff frequency for the TM_m mode.

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Cutoff Modes

The phase constant β_m is calculated as

$$\beta_m = \sqrt{k^2 - \left(\frac{m\pi}{d}\right)^2} \qquad m = 0, 1, 2, 3, \ldots$$

What about when $k^2 < \left(\frac{m\pi}{d}\right)^2$?

$$\beta_m = j\sqrt{\left(\frac{m\pi}{d}\right)^2 - k^2} = j\beta_m''$$

The phase constant β_m becomes imaginary.

$$e^{j\beta_m z} = e^{j(j\beta_m'')z} = e^{-\beta'' z}$$

A mode that is cutoff does not propagate in the longitudinal direction. Instead, it decays, usually very quickly.

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Velocity of the Modes (1 of 2)

The velocity information comes from the phase constant β_m.

$$\beta_m = k_0 n_{\text{eff}}$$

$$\sqrt{k^2 - \left(\frac{m\pi}{d}\right)^2} = k_0 n_{\text{eff}}$$

$$n_{\text{eff}} = \frac{1}{k_0}\sqrt{k^2 - \left(\frac{m\pi}{d}\right)^2} = \sqrt{\left(\frac{k}{k_0}\right)^2 - \left(\frac{m\pi}{k_0 d}\right)^2} = \sqrt{n^2 - \left(\frac{m\pi}{k_0 d}\right)^2}$$

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Velocity of the Modes (1 of 2)

Suppose the waveguide is filled with vacuum (i.e. $n = 1.0$).

$$n_{\text{eff}} = \sqrt{n^2 - \left(\frac{m\pi}{k_0 d}\right)^2} = \sqrt{1 - \left(\frac{m\pi}{k_0 d}\right)^2}$$

Observe that the effective refractive index n_{eff} is always less than 1.

$$n_{\text{eff}} < 1.0 \quad \Rightarrow \quad v > c_0$$

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Characteristic Impedance Z_{TM}

The characteristic impedance Z_{TM} of the TM mode is defined as

$$Z_{\text{TM}} = \frac{E_{0,x}}{H_{0,y}} = -\frac{E_{0,y}}{H_{0,x}}$$

An expression for Z_{TM} is derived by substituting in the expressions for field components.

$$Z_{\text{TM}} = -\frac{E_{0,y}}{H_{0,x}} = -\frac{-\dfrac{j\beta_m}{k_c} A_m \cos\left(\dfrac{m\pi y}{d}\right)}{\dfrac{j\omega\varepsilon}{k_c} A_m \cos\left(\dfrac{m\pi y}{d}\right)} = \frac{\beta_m}{\omega\varepsilon} = \eta \frac{\beta_m}{k}$$

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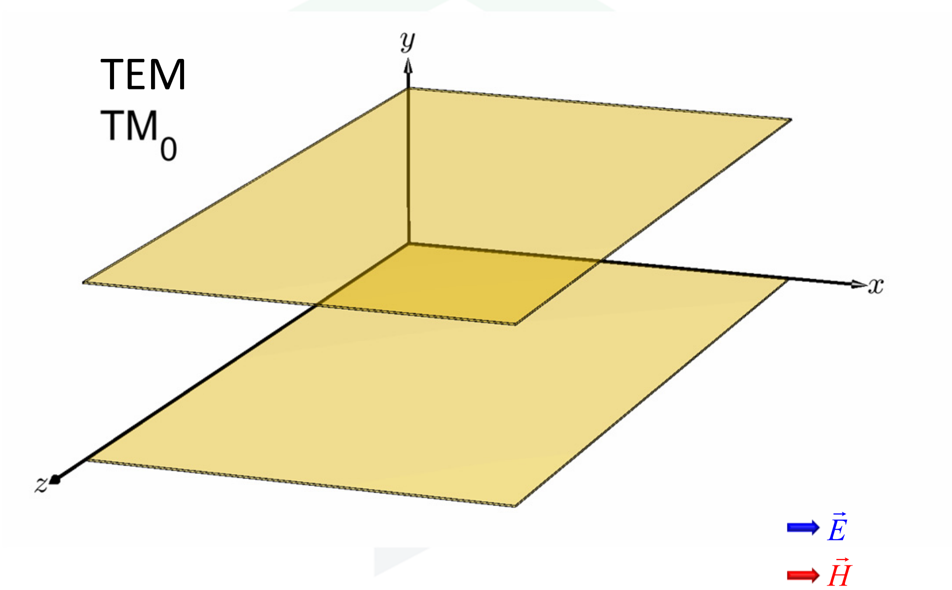
Visualization of the TM Modes

Visualization of TM_0 Mode



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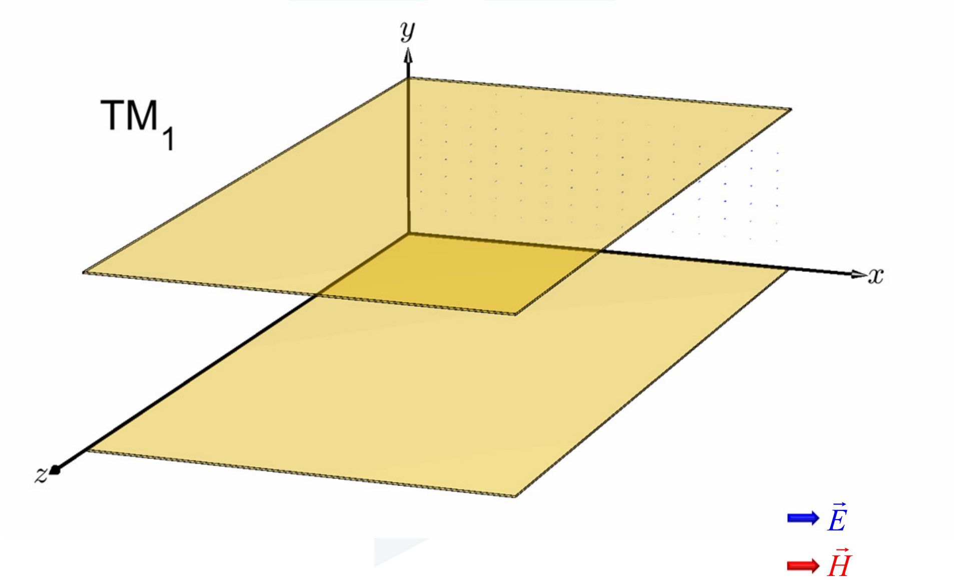
Visualization of TM_1 Mode



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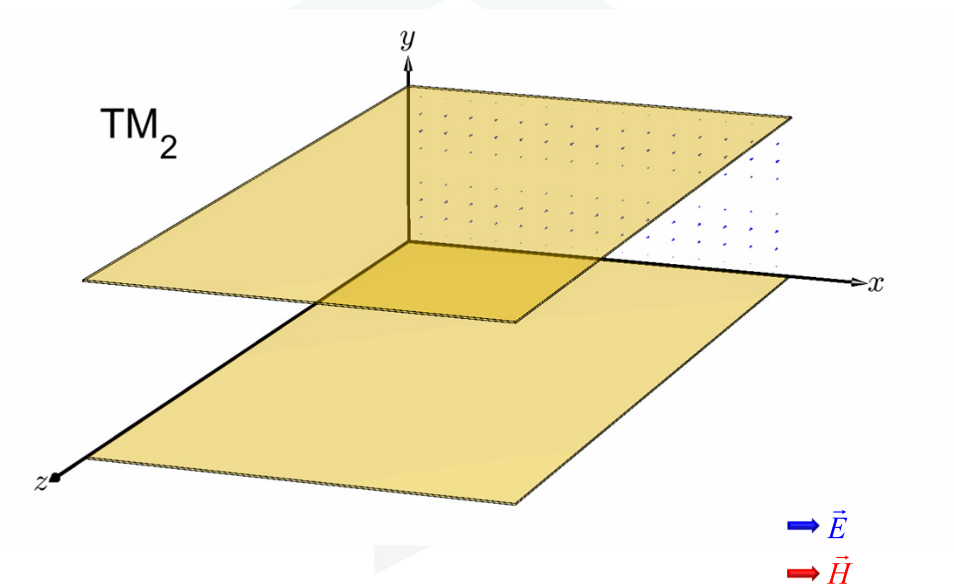
Visualization of TM_2 Mode



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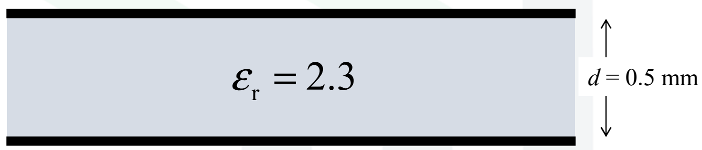
Example

Example #1 (1 of 2)

Given the following parallel plate waveguide...



What is the bandwidth of this waveguide when used as a transmission line?

Example #1 (2 of 2)

When used as a transmission line, it is only the TEM mode that is of interest. The bandwidth is the range of frequencies for which the waveguide supports only the TEM mode.

The cutoff frequencies are the same for the TE and TM modes, so they are essentially checked at the same time.

The second-order modes are TE_1 and TM_1. The bandwidth is simply the cutoff frequency of these modes.

$$f_c(m=1) = \frac{k_c}{2\pi\sqrt{\mu\varepsilon}} = \frac{mc_0}{2d\sqrt{\mu_r\varepsilon_r}} = \frac{(1)(299792458\ \text{m/s})}{2(0.5\ \text{mm})\sqrt{(1.0)(2.3)}} = \boxed{197.6\ \text{GHz}}$$

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Conclusion

Summary of Parallel Plate Waveguide

Parameter	TEM	TM_m $m = 0,1,2,3\ldots$	TE_m $m = 1,2,3\ldots$
k	$\omega\sqrt{\mu\varepsilon}$	$\omega\sqrt{\mu\varepsilon}$	$\omega\sqrt{\mu\varepsilon}$
k_c	0	$m\pi / d$	$m\pi / d$
β	$k = \omega\sqrt{\mu\varepsilon}$	$\sqrt{k^2 - k_c^2}$	$\sqrt{k^2 - k_c^2}$
λ_c	∞	$2\pi/k_c = 2d/n$	$2\pi/k_c = 2d/n$
λ_g	$2\pi / k$	$2\pi / \beta_m$	$2\pi / \beta_m$
v_p	$\omega/k = 1/\sqrt{\mu\varepsilon}$	ω / β_m	ω / β_m
α_d	$k \tan\delta/2$	$k^2 \tan\delta/2\beta_m$	$k^2 \tan\delta/2\beta_m$
α_c	$R_s/\eta d$	$2kR_s/\beta_m\eta d$	$2k_c^2 R_s/k\beta_m\eta d$
E_x	0	0	$(j\omega\mu/k_c) B_m \sin(m\pi y/d) e^{-j\beta_m z}$
E_y	$(-V_0/d) e^{-j\beta z}$	$(-j\beta_m/k_c) A_m \cos(m\pi y/d) e^{-j\beta_m z}$	0
E_z	0	$A_m \sin(m\pi y/d) e^{-j\beta_m z}$	0
H_x	$(-V_0/\eta d) e^{-j\beta z}$	$(j\omega\varepsilon/k_c) A_m \cos(m\pi y/d) e^{-j\beta_m z}$	0
H_y	0	0	$(j\beta_m/k_c) B_m \sin(m\pi y/d) e^{-j\beta_m z}$
H_z	0	0	$B_m \cos(m\pi y/d) e^{-j\beta_m z}$
Z	$\eta d/w$	$\beta_m\eta/k$	$k\eta/\beta_m$

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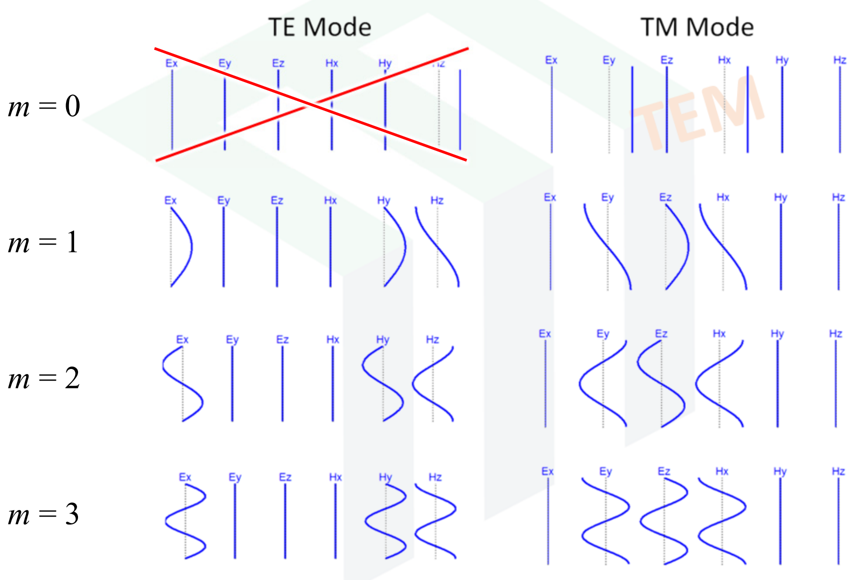
Modes in Parallel Plate Waveguide



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Summary of TM Analysis

Field Solution

$$E_x(x,y,z)=0$$

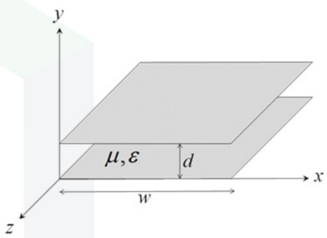
$$E_y(x,y,z)=-\frac{j\beta_m}{k_c}A_m\cos\left(\frac{m\pi y}{d}\right)e^{-j\beta_m z}$$

$$E_z(x,y,z)=A_m\sin\left(\frac{m\pi y}{d}\right)e^{-j\beta_m z}$$

$$H_x(x,y,z)=\frac{j\omega\varepsilon}{k_c}A_m\cos\left(\frac{m\pi y}{d}\right)e^{-j\beta_m z}=\frac{jk}{\eta k_c}A_m\cos\left(\frac{m\pi y}{d}\right)e^{-j\beta_m z}$$

$$H_y(x,y,z)=0$$

$$H_z(x,y,z)=0$$



- TM_0 mode is lowest-order TM mode
- TM_0 mode is the TEM mode

Phase Constant

$$\beta_m=\sqrt{k^2-\left(\frac{m\pi}{d}\right)^2}$$

$m=0,1,2,3,\ldots$

Same equation as for TE

Cutoff Frequency

$$f_{c,m}=\frac{m}{2d\sqrt{\mu\varepsilon}}$$

Same equation as for TE

Characteristic Impedance

$$Z_{TM,m}=\eta\frac{\beta_m}{k}$$

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