Electromagnetics:
Electromagnetic Field Theory

TM Analysis of the Rectangular Metal Waveguide

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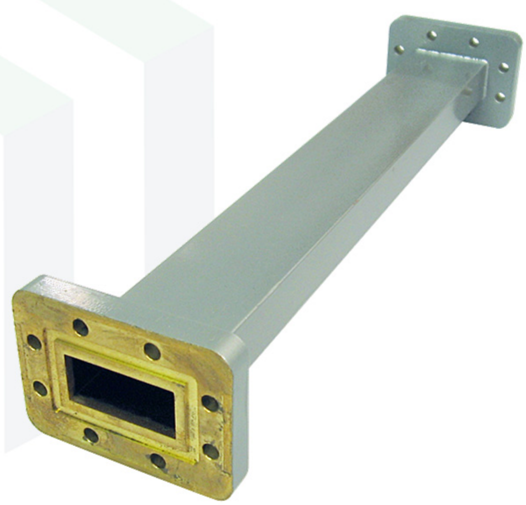
Lecture Outline

- TM Analysis
- Analysis of TM Solution
- Visualization of the Modes
- Example
- Conclusion



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TM Analysis

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Recall TM Analysis

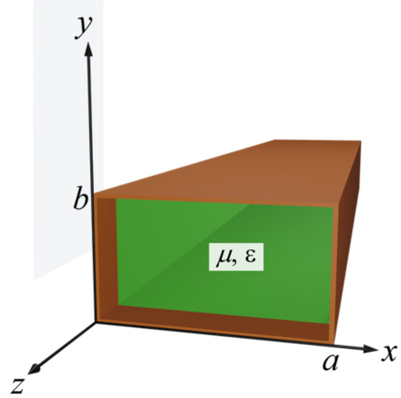
The governing equation for TM analysis is

$$\frac{\partial^2 E_{0,z}}{\partial x^2} + \frac{\partial^2 E_{0,z}}{\partial y^2} - k_c^2 E_{0,z} = 0 \qquad H_{0,z} = 0 \qquad k_c^2 = k^2 - \beta^2$$

After a solution is obtained, the remaining field components are calculated according to

$$H_{0,x} = \frac{j\omega\varepsilon}{k_c^2}\frac{\partial E_{0,z}}{\partial y} \qquad E_{0,x} = -\frac{j\beta}{k_c^2}\frac{\partial E_{0,z}}{\partial x}$$

$$H_{0,y} = -\frac{j\omega\varepsilon}{k_c^2}\frac{\partial E_{0,z}}{\partial x} \qquad E_{0,y} = -\frac{j\beta}{k_c^2}\frac{\partial E_{0,z}}{\partial y}$$



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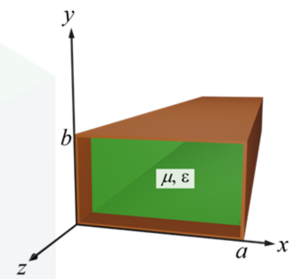
General Form of the Solution

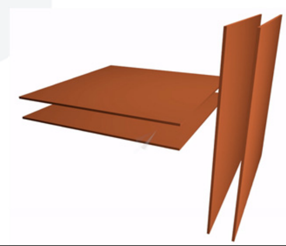
From the geometry of the waveguide, the general form of the solution can be immediately written as

$$E_z(x,y,z) = E_{0,z}(x,y)e^{-j\beta z}$$



Viewing the rectangular waveguide as the combination of two parallel plate waveguides, apply separation of variables to write $E_{0,z}(x,y)$ as the product of two functions.

$$E_{0,z}(x,y) = X(x)Y(y)$$



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Separation of Variables (1 of 3)

The solution is written as the product of two 1D functions, $X(x)$ and $Y(y)$. Substitute this solution back into the differential equation.

$E_{0,z}(x,y) = X(x)Y(y)$

$$\frac{\partial^2 E_{0,z}}{\partial x^2} + \frac{\partial^2 E_{0,z}}{\partial y^2} - k_c^2 E_{0,z} = 0$$

$$\Downarrow$$

$$\frac{\partial^2 XY}{\partial x^2} + \frac{\partial^2 XY}{\partial y^2} - k_c^2 XY = 0$$

To be compact, drop the (x) and (y) notation.

$$\frac{\partial^2 X}{\partial x^2}Y + X\frac{\partial^2 Y}{\partial y^2} - k_c^2 XY = 0$$

Move $Y(y)$ out of the $\partial^2/\partial x^2$ operation and $X(x)$ out of the $\partial^2/\partial y^2$ operation.

$$\frac{1}{X}\frac{d^2 X}{dx^2} + \frac{1}{Y}\frac{d^2 Y}{dy^2} - k_c^2 = 0$$

Divide both sides by XY. The derivatives become ordinary because $X(x)$ and $Y(y)$ have only one independent variable each.

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Separation of Variables (2 of 3)

First, attention is focused on the x dependence in the differential equation.

$$\frac{1}{X}\frac{d^2X}{dx^2}+\underbrace{\frac{1}{Y}\frac{d^2Y}{dy^2}}_{-k_x^2}-k_c^2=0$$

This group of terms has no x dependence so it can be treated as a constant. This definition of k_x lets the differential equation be written as a wave equation.

$$\frac{d^2X}{dx^2}-k_x^2X=0$$

Second, attention is focused on the y dependence in the differential equation.

$$\underbrace{\frac{1}{X}\frac{d^2X}{dx^2}-k_c^2}_{-k_y^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}=0$$

This group of terms has no y dependence so it can be treated as a constant. This definition of k_y lets the differential equation be written as a wave equation.

$$\frac{d^2Y}{dy^2}-k_y^2Y=0$$

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Separation of Variables (3 of 3)

It is necessary that the sum of the two new differential equations be the original differential equation.

$$\frac{d^2X}{dx^2}-k_x^2X=0$$
$$\frac{d^2Y}{dy^2}-k_y^2Y=0$$

$$\Rightarrow$$

$$\frac{1}{X}\frac{d^2X}{dx^2}-k_x^2=0$$
$$+\ \frac{1}{Y}\frac{d^2Y}{dy^2}-k_y^2=0$$
$$\overline{\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}-k_x^2-k_y^2=0}$$

Recall the original differential equation

$$\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}-k_c^2=0$$

The original differential equation is obtained if

$$k_c^2=k_x^2+k_y^2$$

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General Solution

There are now two differential equations to solve.

$$\frac{d^2X}{dx^2} - k_x^2 X = 0 \qquad \frac{d^2Y}{dy^2} - k_y^2 Y = 0$$

These are essentially the same differential equation so their solution has the same general form.

$$\frac{d^2X}{dx^2} - k_x^2 X = 0 \quad \rightarrow \quad X(x) = A\cos(k_x x) + B\sin(k_x x)$$

Parallel plate waveguide along x

$$\frac{d^2Y}{dy^2} - k_y^2 Y = 0 \quad \rightarrow \quad Y(y) = C\cos(k_y y) + D\sin(k_y y)$$

Parallel plate waveguide along y

The overall solution is the product of $X(x)$ and $Y(y)$.

$$E_{0,z}(x,y) = X(x)Y(y) = \left[A\cos(k_x x) + B\sin(k_x x)\right]\left[C\cos(k_y y) + D\sin(k_y y)\right]$$

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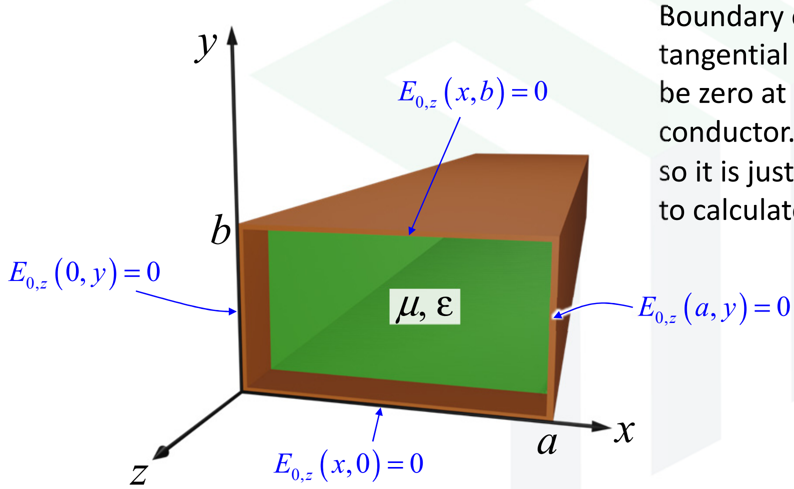
Electromagnetic Boundary Conditions



Boundary conditions require that the tangential component of the electric field be zero at the boundary with a perfect conductor. $E_{0,z}$ is tangential to all interfaces, so it is just used directly. There is no need to calculate any other field components.

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Apply Boundary Conditions (1 of 2)

At the $x = 0$ boundary,

$$0 = E_{0,z}(0, y)$$
$$= [A\cos(k_x 0) + \cancel{B\sin(k_x 0)}]\,[C\cos(k_y y) + D\sin(k_y y)]$$
$$= A[C\cos(k_y y) + D\sin(k_y y)] \longrightarrow A = 0$$

At the $x = a$ boundary,

$$0 = E_{0,z}(a, y)$$
$$= [B\sin(k_x a)]\,[C\cos(k_y y) + D\sin(k_y y)]$$

$B = 0$ leads to a trivial solution. It must be the $\sin(k_x a)$ term that enforces the boundary condition.

$$0 = \sin(k_x a) \longrightarrow k_x a = m\pi \quad m = 1,2,\ldots$$

$m = 0$ leads to a trivial solution.
This will be shown later.

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Apply Boundary Conditions (2 of 2)

At the $y = 0$ boundary,

$$0 = E_{0,z}(x, 0)$$
$$= [A\cos(k_x x) + B\sin(k_x x)]\,[C\cos(k_y 0) + \cancel{D\sin(k_y 0)}]$$
$$= [A\cos(k_x x) + B\sin(k_x x)]C \longrightarrow C = 0$$

At the $y = b$ boundary,

$$0 = E_{0,z}(x, b)$$
$$= [A\cos(k_x x) + B\sin(k_x x)][D\sin(k_y b)]$$

$D = 0$ leads to a trivial solution. It must be the $\sin(k_y b)$ term that enforces the boundary condition.

$$0 = \sin(k_y b) \longrightarrow k_y b = n\pi \quad n = 1,2,\ldots$$

$n = 0$ leads to a trivial solution.
This will be shown later.

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Revised Solution for $E_{0,z}$

It was determined that $A = C = 0$ so the expression for $E_{0,z}(x,y)$ becomes

$$E_{0,z}(x,y) = \boxed{BD}\sin(k_x x)\sin(k_y y)$$

The product BD is written as a single constant B_{mn}.

$$E_{0,z}(x,y) = B_{mn}\sin(k_x x)\sin(k_y y)$$

Also, recall the conditions for k_x and k_y.

$$k_x a = m\pi \quad \rightarrow \quad k_x = \frac{m\pi}{a} \qquad\qquad k_y b = n\pi \quad \rightarrow \quad k_y = \frac{n\pi}{b}$$

$$E_{0,z}(x,y) = B_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

* Note: Neither m nor n can be zero or the entire solution will be zero.

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Entire Solution (1 of 2)

The final expression for $E_{0,z}(x,y)$ is

$$E_{0,z}(x,y) = B_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \qquad\qquad H_{0,z}(x,y) = 0$$

From this, the other field components are

$$E_{0,x}(x,y) = -\frac{j\beta_{mn} m\pi}{k_c^2 a} B_{mn}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$E_{0,y}(x,y) = -\frac{j\beta_{mn} n\pi}{k_c^2 b} B_{mn}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

$$H_{0,x}(x,y) = \frac{j\omega\varepsilon n\pi}{k_c^2 b} B_{mn}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

$$H_{0,y}(x,y) = -\frac{j\omega\varepsilon m\pi}{k_c^2 a} B_{mn}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

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Entire Solution (2 of 2)

The overall electric and magnetic fields at any position are

$$E_x(x,y,z) = -\frac{j\beta_{mn} m\pi}{k_c^2 a} B_{mn} \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-j\beta_{mn}z}$$

$$E_y(x,y,z) = -\frac{j\beta_{mn} n\pi}{k_c^2 b} B_{mn} \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-j\beta_{mn}z}$$

$$E_z(x,y,z) = B_{mn} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-j\beta_{mn}z}$$

$$H_x(x,y,z) = \frac{j\omega\varepsilon n\pi}{k_c^2 b} B_{mn} \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-j\beta_{mn}z}$$

$$H_y(x,y,z) = -\frac{j\omega\varepsilon m\pi}{k_c^2 a} B_{mn} \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-j\beta_{mn}z}$$

$$H_z(x,y,z) = 0$$

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Analysis of TM Solution

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Phase Constant β_{mn}

Recall the cutoff wave number

$$k_c^2 = k_x^2 + k_y^2$$

After analyzing the boundary conditions, this expression can be written as

$$k_{c,mn}^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

The phase constant β_{mn} is therefore

$$k_{c,mn}^2 = k^2 - \beta_{mn}^2$$

$$\beta_{mn}^2 = k^2 - k_{c,mn}^2$$

$$\boxed{\beta_{mn} = \sqrt{k^2 - k_{c,mn}^2} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}}$$

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Cutoff Frequency $f_{c,mn}$

Recall the expression for the phase constant

$$\beta_{mn} = \sqrt{k^2 - k_{c,mn}^2}$$

The phase constant must be a real number for a guided mode. This requires

$$k > k_{c,mn}$$

Any time $k < k_{c,mn}$, the mode is cutoff and not supported by the waveguide. From this, the cutoff frequency $f_{c,mn}$ can be derived as

$$k > k_{c,mn}$$

$$\omega\sqrt{\mu\varepsilon} > k_{c,mn}$$

$$2\pi f_{c,mn}\sqrt{\mu\varepsilon} = k_{c,mn}$$

$$\boxed{f_{c,mn} = \frac{k_{c,mn}}{2\pi\sqrt{\mu\varepsilon}} = \frac{c_0}{2\pi\sqrt{\mu_r\varepsilon_r}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}}$$

This is the same equation as for the TE modes.

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Characteristic Impedance, Z_{TM}

The characteristic impedance Z_{TM} for the TM mode is

$$Z_{\text{TM}} = \frac{E_x}{H_y} = \frac{-\dfrac{j\beta_{mn} m\pi}{k_c^2 a} B_{mn} \cos\left(\dfrac{m\pi x}{a}\right) \sin\left(\dfrac{n\pi y}{b}\right) e^{-j\beta z}}{-\dfrac{j\omega\varepsilon m\pi}{k_c^2 a} B_{mn} \cos\left(\dfrac{m\pi x}{a}\right) \sin\left(\dfrac{n\pi y}{b}\right) e^{-j\beta z}} = \frac{\beta_{mn}}{\omega\varepsilon} = \frac{\beta_{mn}\eta}{k}$$

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Cutoff for First-Order TM Mode (1 of 2)

The cutoff frequency for the TM$_{mn}$ mode was found to be

$$f_{c,mn} = \frac{1}{2\pi\sqrt{\mu\varepsilon}} \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

Note, it is not possible to have $n = 0$ or $m = 0$ for the TM mode. So…

The TM$_{00}$ mode does not exist.
The TM$_{01}$ mode does not exist.
The TM$_{10}$ mode does not exist.
The TM$_{02}$ mode does not exist.
The TM$_{20}$ mode does not exist.
The TM$_{03}$ mode does not exist.
The TM$_{30}$ mode does not exist.

etc.

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Cutoff for First-Order TM Mode (2 of 2)

What combination of m and n minimizes f_c?

$$f_{c,mn} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

The TM_{11} mode will have the lowest cutoff frequency.

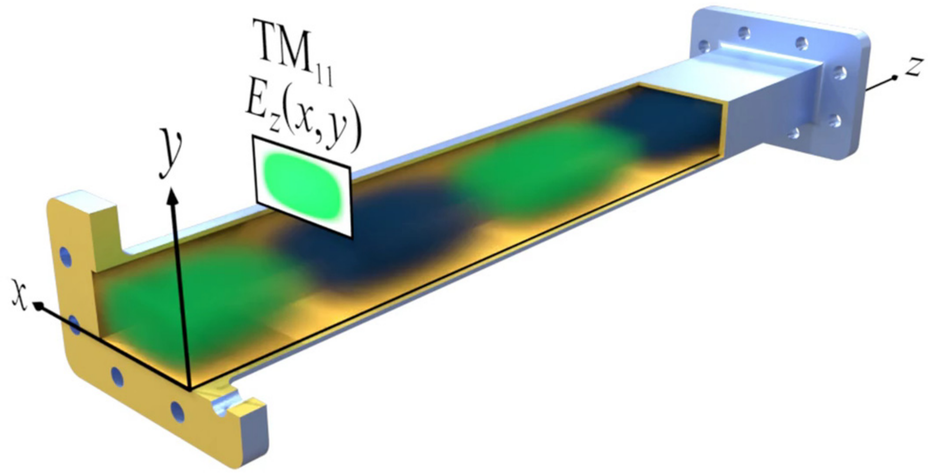
$$m = 1, n = 1$$

$$f_{c1} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{1\cdot\pi}{a}\right)^2 + \left(\frac{1\cdot\pi}{b}\right)^2} = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2}$$

CAUTION: It cannot yet be said that the TM_{11} is the fundamental mode because the TE modes have not been checked.

Visualization of the TM Modes

Visualization of E_z for the TM_{11} Mode



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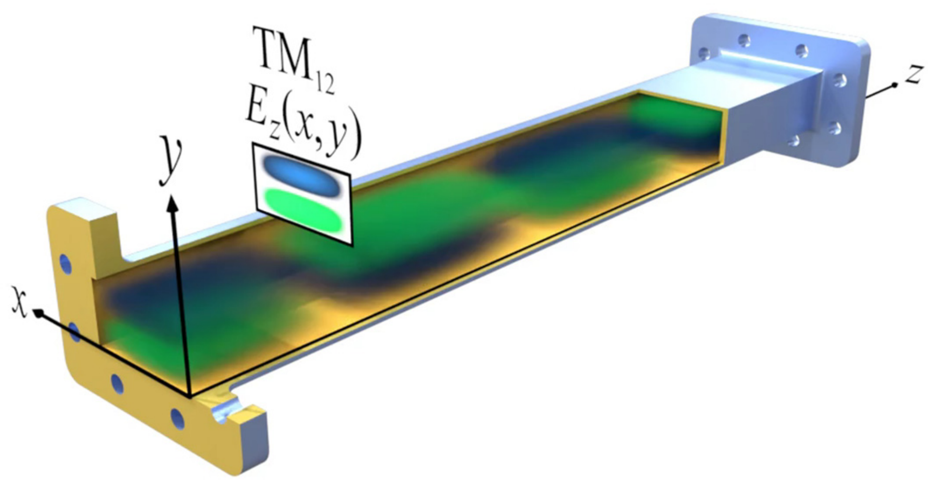
Visualization of E_z for the TM_{12} Mode



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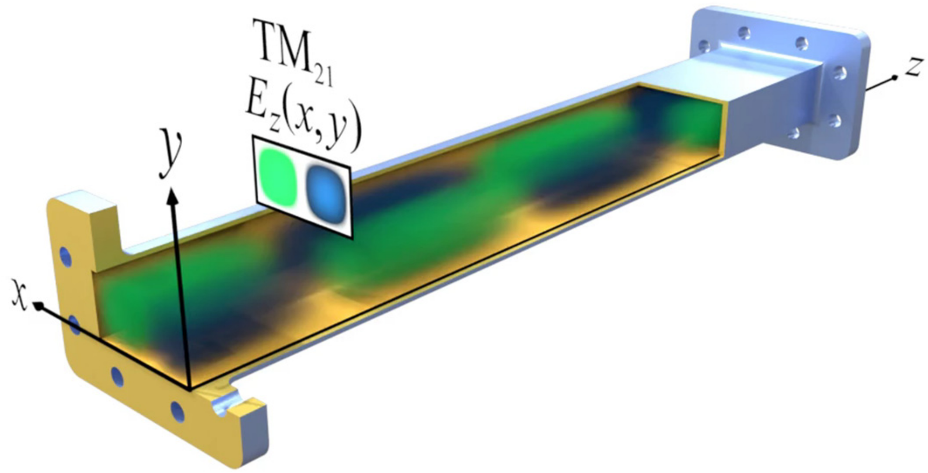
Visualization of E_z for the TM_{21} Mode



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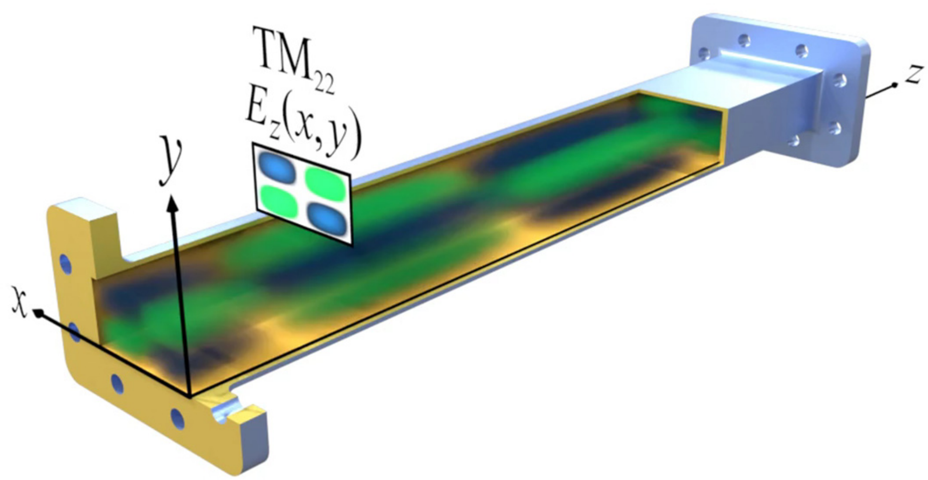
Visualization of E_z for the TM_{22} Mode



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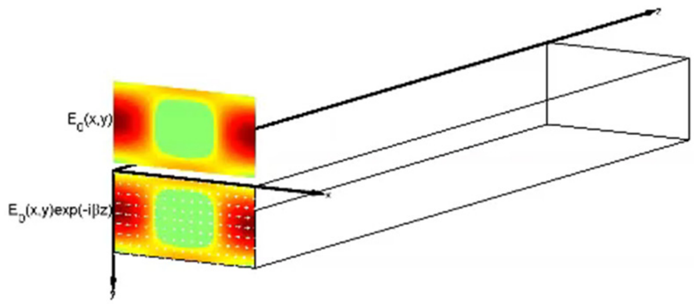
Animation of TM_{11}



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Example

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Example – TM Mode Analysis (1 of 3)

Given an air-filled rectangular waveguide with a = 3 cm and b = 2 cm.

What is the cutoff frequency of the waveguide?

$$f_{c1} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2}$$

$$= \frac{1}{2\pi\sqrt{\mu_0\mu_r\varepsilon_0\varepsilon_r}}\sqrt{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2}$$

$$= \frac{1}{2\pi\sqrt{\mu_0\varepsilon_0}\sqrt{\mu_r\varepsilon_r}}\sqrt{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2}$$

$$= \frac{c_0}{2\sqrt{\mu_r\varepsilon_r}}\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2}$$

$$= \frac{299792458 \text{ m/s}}{2\sqrt{(1.0)(1.0)}}\sqrt{\left(\frac{1}{0.03 \text{ m}}\right)^2 + \left(\frac{1}{0.02 \text{ m}}\right)^2} = \boxed{9.0 \text{ GHz}}$$

Recall that $c_0 = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$

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Example – TM Mode Analysis (2 of 3)

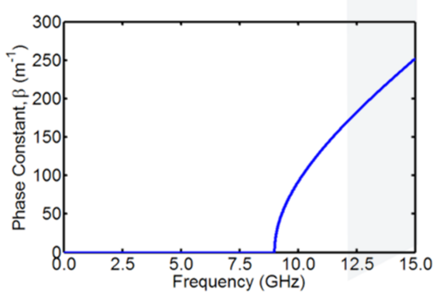
Plot the phase constant and effective refractive index for the first-order mode from DC up to 15 GHz.

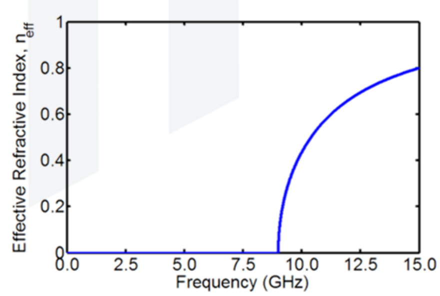
The phase constant is calculated as:

$$\beta_{mn} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} \quad \rightarrow \quad \text{TM}_{11}\text{:} \;\; \beta = \sqrt{\left(\frac{2\pi f}{c_0}\right)^2 - \left(\frac{\pi}{a}\right)^2}$$

The effective refractive index is calculated as:

$$\beta = k_0 n_{\text{eff}} \quad \rightarrow \quad n_{\text{eff}} = \frac{\beta}{\frac{2\pi f}{c_0}} = \beta\frac{c_0}{2\pi f}$$





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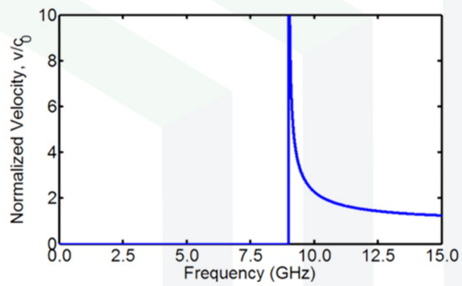
Example – TM Mode Analysis (3 of 3)

Plot the velocity of the modes as a function of frequency.

$$v = \frac{c_0}{n_{\text{eff}}}$$



Are the modes travelling faster than the speed of light?

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Conclusion

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Summary of TM Analysis

Field Solution

$$E_x(x,y,z) = -\frac{j\beta_{mn} m\pi}{k_c^2 a} B_{mn} \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-j\beta_{mn}z}$$

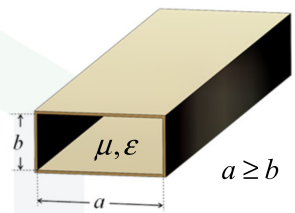
$$E_y(x,y,z) = -\frac{j\beta_{mn} n\pi}{k_c^2 b} B_{mn} \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-j\beta_{mn}z}$$

$$E_z(x,y,z) = B_{mn} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-j\beta_{mn}z}$$

$$H_x(x,y,z) = \frac{j\omega\varepsilon n\pi}{k_c^2 b} B_{mn} \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-j\beta_{mn}z}$$

$$H_y(x,y,z) = -\frac{j\omega\varepsilon m\pi}{k_c^2 a} B_{mn} \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-j\beta_{mn}z}$$

$$H_z(x,y,z) = 0$$



- $m \neq 0$ and $n \neq 0$, so TM_{00}, TM_{01}, TM_{02}, TM_{10}, TM_{20}, etc. are not supported modes.
- TM_{11} is the lowest order TM mode

Phase Constant

$$\beta_{mn} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

Same equation as for TE

Cutoff Frequency

$$f_{c,mn} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

Same equation as for TE

Characteristic Impedance

$$Z_{TM,mn} = \frac{\eta\beta_{mn}}{k}$$

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