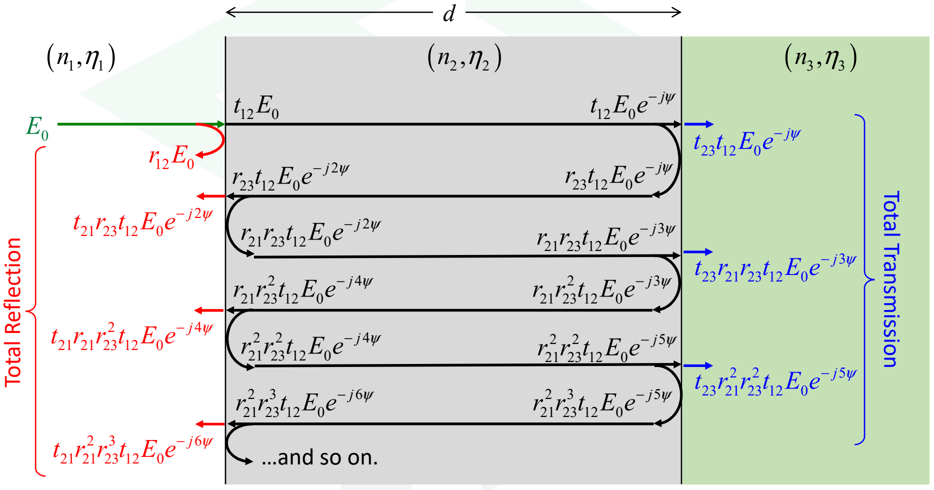
Electromagnetics:

Electromagnetic Field Theory

Scattering from a Dielectric Slab

Analysis of a Dielectric Slab



$$r_{12}=\frac{\eta_2-\eta_1}{\eta_2+\eta_1} \qquad t_{12}=1-r_{12}$$

$$r_{21}=\frac{\eta_1-\eta_2}{\eta_1+\eta_2}=-r_{12} \qquad t_{21}=1+r_{12}$$

$$\psi=k_0n_2d$$

$$r_{23}=\frac{\eta_3-\eta_2}{\eta_3+\eta_2}$$

$$t_{23}=1-r_{23}$$

Overall Reflection Coefficient r (1 of 3)

The overall reflection from the slab is the sum of all the individual reflected waves.

$$r = r_{12} + \underbrace{t_{21}r_{23}t_{12}e^{-j2\psi} + t_{21}r_{21}r_{23}^{2}t_{12}e^{-j4\psi} + t_{21}r_{21}^{3}r_{23}^{3}t_{12}e^{-j6\psi} + \cdots}$$

All of these terms arise due to multiple reflections within the slab. They can be written as a summation.

$$\sum_{n=0}^{\infty} t_{21}r_{21}^{n}r_{23}^{n+1}t_{12}e^{-j2(n+1)\psi}$$

$$r_{23}t_{21}t_{12}e^{-j2\psi}\sum_{n=0}^{\infty}\left(r_{21}r_{23}e^{-j2\psi}\right)^{n}$$

Factor out the term $r_{23}t_{21}t_{12}e^{-j2\psi}$.

Now put the summation back into the expression for overall reflection r.

$$r = r_{12} + r_{23}t_{21}t_{12}e^{-j2\psi}\sum_{n=0}^{\infty}\left(r_{21}r_{23}e^{-j2\psi}\right)^{n}$$

Overall Reflection Coefficient r (2 of 3)

Recall the closed-form expression for a geometric series.

$$\sum_{n=0}^{\infty} ax^{n} = \frac{a}{1-x} \qquad \text{for } |x| < 1$$

The summation in the expression for overall reflection r is a geometric series that can be written in closed form.

$$r = r_{12} + r_{23}t_{21}t_{12}e^{-j2\psi}\sum_{n=0}^{\infty}\left(r_{21}r_{23}e^{-j2\psi}\right)^{n}$$

Expression from last slide. Observe that $\left|r_{21}r_{23}e^{-j2\psi}\right| < 1$.

$$= r_{12} + r_{23}t_{21}t_{12}e^{-j2\psi}\frac{1}{1-r_{21}r_{23}e^{-j2\psi}}$$

Replace summary with closed form expression for a geometry series.

$$= r_{12} + \frac{r_{23}t_{21}t_{12}e^{-j2\psi}}{1-r_{21}r_{23}e^{-j2\psi}}$$

Adjust position of terms to make equation compact.

Overall Reflection Coefficient r (3 of 3)

Recall how the local reflection and transmission parameters were related.

$$r_{21} = -r_{12} \qquad t_{12} = 1 - r_{12} \qquad t_{21} = 1 + r_{12} \qquad t_{23} = 1 - r_{23}$$

This lets r be expressed just in terms of r_{12}, r_{23} and ψ.

$$r = r_{12} + \frac{r_{23} t_{21} t_{12} e^{-j2\psi}}{1 - r_{21} r_{23} e^{-j2\psi}}$$

Expression from last slide.

$$= r_{12} + \frac{r_{23}\left(1 + r_{12}\right)\left(1 - r_{12}\right) e^{-j2\psi}}{1 - \left(-r_{12}\right) r_{23} e^{-j2\psi}}$$

Replace t_{21}, t_{12} and r_{21}.

$$= \frac{r_{12}\left(1 + r_{12} r_{23} e^{-j2\psi}\right)}{1 + r_{12} r_{23} e^{-j2\psi}} + \frac{r_{23} e^{-j2\psi} - r_{12}^2 r_{23} e^{-j2\psi}}{1 + r_{12} r_{23} e^{-j2\psi}} \quad \rightarrow \quad \boxed{r = \frac{r_{12} + r_{23} e^{-j2\psi}}{1 + r_{12} r_{23} e^{-j2\psi}}}$$

Overall Transmission Coefficient t (1 of 2)

The overall transmission through the slab is the sum of all the individual transmitted waves.

$$t = t_{23} t_{12} e^{-j\psi} + t_{23} r_{21} r_{23} t_{12} e^{-j3\psi} + t_{23} r_{21}^2 r_{23}^2 t_{12} e^{-j5\psi} + \cdots$$

This can be written as a summation.

$$t = \sum_{n=0}^{\infty} t_{23} r_{21}^n r_{23}^n t_{12} e^{-j(2n+1)\psi}$$

Now factor out $t_{23} t_{12} e^{-j\psi}$ from the summation to get the form of a geometric series.

$$t = t_{23} t_{12} e^{-j\psi} \sum_{n=0}^{\infty} \left(r_{21} r_{23} e^{-j2\psi}\right)^n$$

The summation in this expression is a geometric series and can be written in closed form.

$$t = t_{23} t_{12} e^{-j\psi} \frac{1}{1 - r_{21} r_{23} e^{-j2\psi}}$$

Overall Transmission Coefficient t (2 of 2)

Recall how the local reflection and transmission parameters were related.

$$r_{21} = -r_{12} \qquad t_{12} = 1 - r_{12} \qquad t_{21} = 1 + r_{12} \qquad t_{23} = 1 - r_{23}$$

This let's t be expressed just in terms of r_{12}, r_{23} and ψ.

$$t = \frac{t_{23} t_{12} e^{-j\psi}}{1 - r_{21} r_{23} e^{-j2\psi}}$$

$$= \frac{(1-r_{23})(1-r_{12}) e^{-j\psi}}{1-(-r_{12}) r_{23} e^{-j2\psi}} \quad \rightarrow \quad \boxed{t = \frac{(1-r_{23})(1-r_{12}) e^{-j\psi}}{1 + r_{12} r_{23} e^{-j2\psi}}}$$

Relation Between r and t

Solve the expressions for r and t for $(1 + r_{12} r_{23} e^{-j2\psi})$.

$$r = \frac{r_{12} + r_{23} e^{-j2\psi}}{1 + r_{12} r_{23} e^{-j2\psi}} \qquad t = \frac{(1-r_{23})(1-r_{12}) e^{-j\psi}}{1 + r_{12} r_{23} e^{-j2\psi}}$$

$$1 + r_{12} r_{23} e^{-j2\psi} = \frac{r_{12} + r_{23} e^{-j2\psi}}{r} \qquad 1 + r_{12} r_{23} e^{-j2\psi} = \frac{(1-r_{23})(1-r_{12}) e^{-j\psi}}{t}$$

The expressions on the right-hand side of the above equations must be equal.

$$\frac{(1-r_{23})(1-r_{12}) e^{-j\psi}}{t} = \frac{r_{12} + r_{23} e^{-j2\psi}}{r}$$

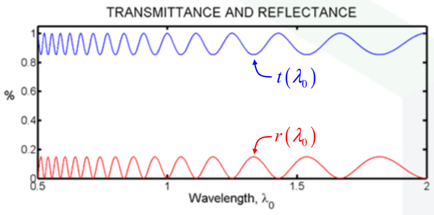
The relation between r and t is therefore

$$\frac{t}{r} = \frac{(1-r_{23})(1-r_{12})}{r_{12} + r_{23} e^{-j\psi}}$$

Note: This is NOT the same relation that was derived for a single interface. This is for a slab.

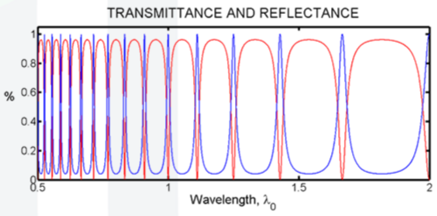
Plots of r and t

Small Reflections
(low finesse)



The response resembles a cosine function and is usually approximated as such.

Large Reflections
(high finesse)



The response resembles a comb filter.

Low Finesse (1 of 2)

To understand low finesse, assume the slab is symmetric (i.e. $r_{12} = -r_{23}$) and that the reflection coefficients are small (< ~5%).

$$r=\frac{r_{12}+r_{23}e^{-j2\psi}}{1+r_{12}r_{23}e^{-j2\psi}}=\frac{r_{12}+\left(-r_{12}\right)e^{-j2\psi}}{1+r_{12}\left(-r_{12}\right)e^{-j2\psi}}=\frac{r_{12}\left(1-e^{-j2\psi}\right)}{\boxed{1-r_{12}^{2}e^{-j2\psi}}}$$

For small reflections,
$1-r_{12}^{2}e^{-j2\psi}\approx 1$

$$t=\frac{\left(1-r_{23}\right)\left(1-r_{12}\right)e^{-j\psi}}{1+r_{12}r_{23}e^{-j2\psi}}=\frac{\left(1-\left(-r_{12}\right)\right)\left(1-r_{12}\right)e^{-j\psi}}{1+r_{12}\left(-r_{12}\right)e^{-j2\psi}}=\frac{\left(1+r_{12}\right)\left(1-r_{12}\right)e^{-j\psi}}{\boxed{1-r_{12}^{2}e^{-j2\psi}}}$$

The expressions for r and t reduce to

$$\boxed{r=r_{12}\left(1-e^{-j2\psi}\right)}$$

$$\boxed{t=\left(1-r_{12}^{2}\right)e^{-j\psi}}$$

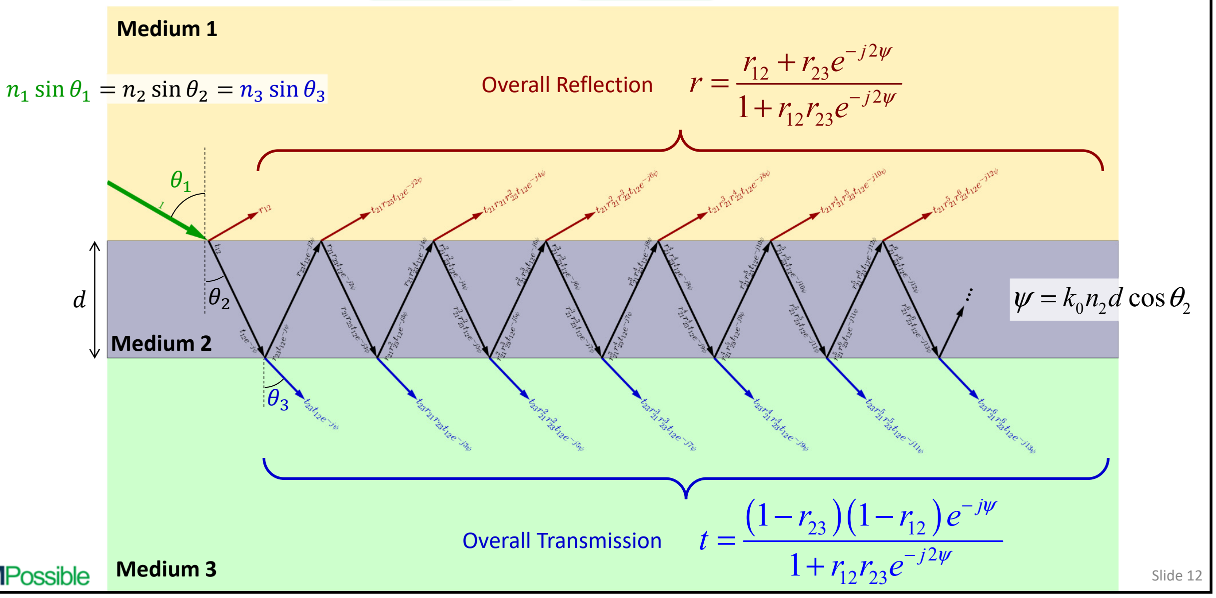
Low Finesse (2 of 2)

The magnitude of r gives the sine wave response that was expected.

$r = r_{12}\left(1 - e^{-j2\psi}\right)$	Expression from last slide.
$= r_{12}e^{-j\psi}\left(e^{-j\psi} - e^{-j\psi}\right)$	Factor out $e^{-j\psi}$.
$= r_{12}e^{-j\psi}\left(j2\sin\psi\right)$	Recognize $\sin\psi = (e^{j\psi} - e^{-j\psi})/j2$.
$= j2r_{12}e^{-j\psi}\sin\psi$	Reorder terms.
$\lvert r\rvert = \lvert j2r_{12}e^{-j\psi}\sin\psi\rvert$	Magnitude of previous equation.
$= \underbrace{\lvert j\rvert}_{1}\underbrace{\lvert 2\rvert}_{2}\underbrace{\lvert r_{12}\rvert}_{\lvert r_{12}\rvert}\underbrace{\lvert e^{-j\psi}\rvert}_{1}\underbrace{\lvert\sin\psi\rvert}_{\sin\psi}$	Examine magnitude of each term.

$$\boxed{\lvert r\rvert = 2\lvert r_{12}\rvert\sin\psi}$$

Oblique Incidence



$$n_1\sin\theta_1 = n_2\sin\theta_2 = n_3\sin\theta_3$$

Overall Reflection $$r = \frac{r_{12} + r_{23}e^{-j2\psi}}{1 + r_{12}r_{23}e^{-j2\psi}}$$

$$\psi = k_0 n_2 d\cos\theta_2$$

Overall Transmission $$t = \frac{\left(1 - r_{23}\right)\left(1 - r_{12}\right)e^{-j\psi}}{1 + r_{12}r_{23}e^{-j2\psi}}$$

Generalizations (1 of 2)

Normal incidence, no loss: $\psi = k_0 n_2 d$

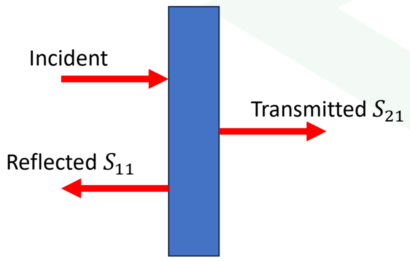
Oblique incidence, no loss: $\psi = k_0 n_2 d \cos\theta_2$

Normal incidence, lossy: $\psi = k_0 \tilde{n}_2 d, \ \tilde{n}_2 = n_\text{o} - j\kappa$

Oblique incidence, lossy: $\psi = k_0 \tilde{n}_2 d \cos\theta_2, \ \tilde{n}_2 = n_\text{o} - j\kappa$

Generalizations (2 of 2)

Suppose the slab is symmetric.



Conditions for Symmetric Slab

$$r = r_{12} = -r_{23}$$
$$t = e^{-j\psi}$$

$$\psi = k_0 n_2 d$$

Scattering Parameters

$$S_{11} = \frac{\left(1-t^2\right)r}{1-r^2t^2} \qquad S_{21} = \frac{\left(1-r^2\right)t}{1-r^2t^2}$$

Example #1: Do Windows Block Wifi? (1 of 2)

Windows are typically made of fused silica (n = 1.52) and are around 3 mm thick.

Solution

Transmission through a slab of dielectric is calculated using

$$t = \frac{(1-r_{23})(1-r_{12})e^{-j\psi}}{1+r_{12}r_{23}e^{-j2\psi}}$$

The parameters in this equation are

$$r_{12} = \frac{\eta_2-\eta_1}{\eta_2+\eta_1} = \frac{n_1-n_2}{n_1+n_2} = \frac{1.0-1.52}{1.0+1.52} = -0.2063$$

$$r_{23} = \frac{\eta_3-\eta_2}{\eta_3+\eta_2} = \frac{n_2-n_3}{n_2+n_3} = \frac{1.52-1.0}{1.52+1.0} = +0.2063$$

$$\psi = k_0 nd = \frac{2\pi f}{c_0} nd = \frac{2\pi\left(2.4\times10^9 \text{ Hz}\right)}{\left(3.0\times10^8 \text{ m/s}\right)}(1.52)(0.003 \text{ m}) = 0.2292$$

Example #1: Do Windows Block Wifi? (2 of 2)

Windows are typically made of fused silica (n = 1.52) and are around 3 mm thick.

Solution cont'd

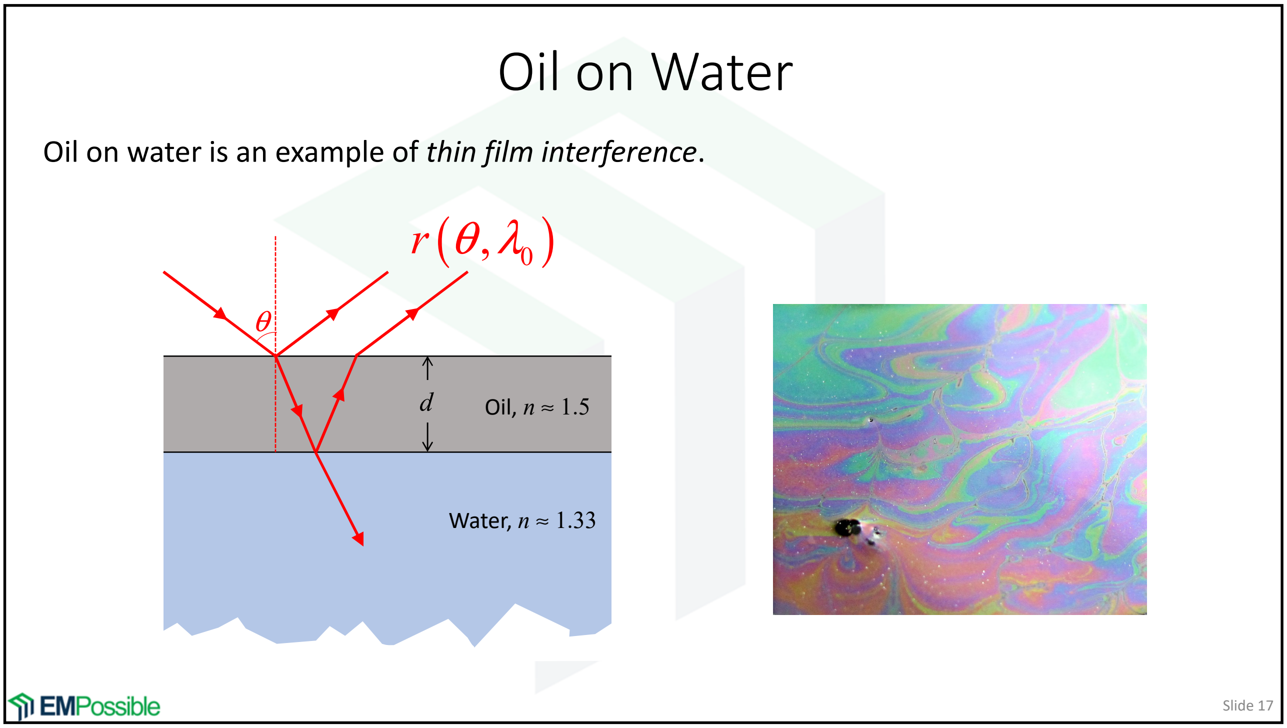
Substituting our values into the transmission equation gives

$$t = \frac{(1-0.2063)(1+0.2063)e^{-j0.2292}}{1+(-0.2063)(0.2063)e^{-j2(0.2292)}} = 0.9646 - j0.2450$$

Total power transmitted is

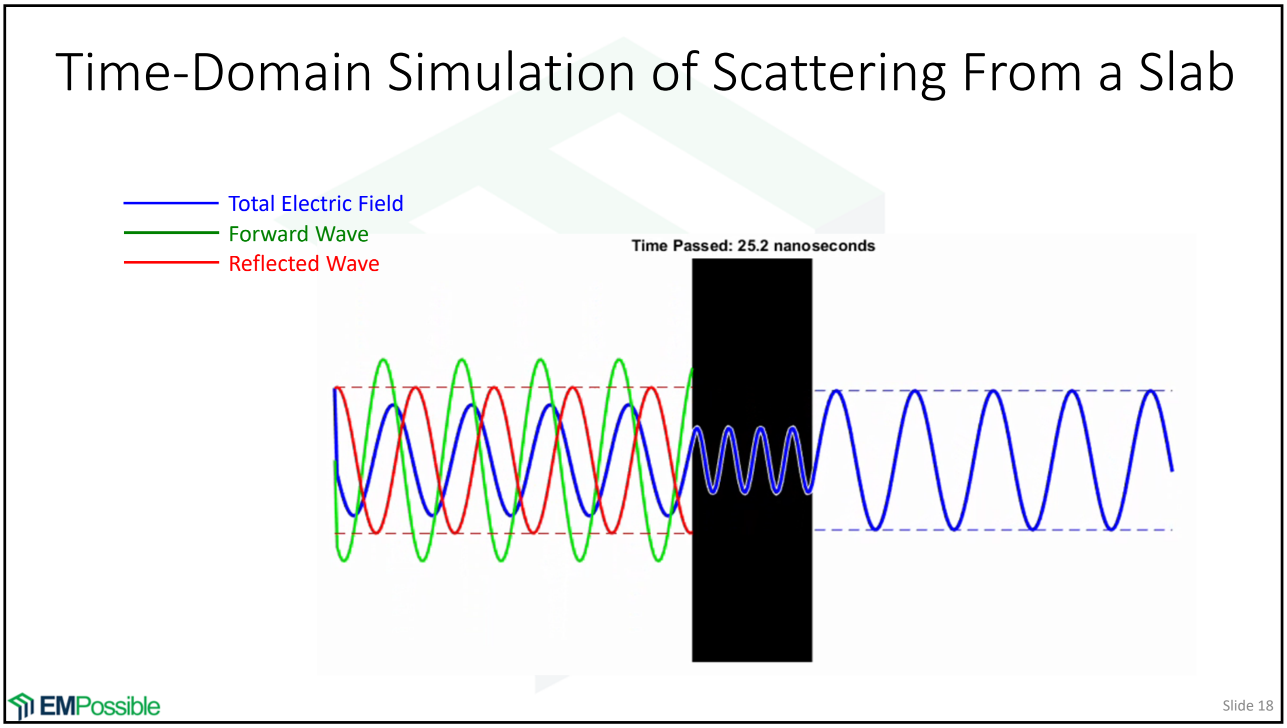
$$T = |t|^2 = |0.9646 - j0.2450|^2 = 99.52\%$$

CONCLUSION → Windows do almost nothing to block Wifi.



Oil on Water

Oil on water is an example of *thin film interference*.



Time-Domain Simulation of Scattering From a Slab