Electromagnetics:
Electromagnetic Field Theory

Transmission Line Equations

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Lecture Outline

- Transmission Line Equations
- Transmission Line Wave Equations

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Transmission Line Equations

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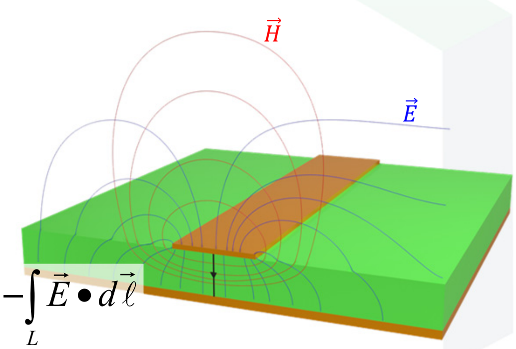
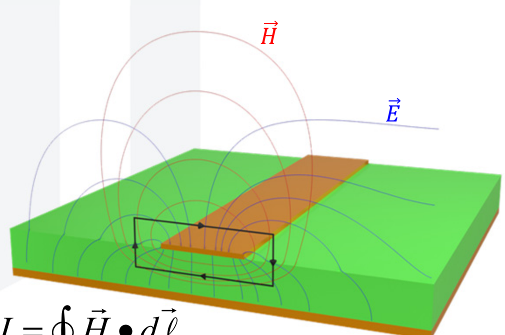
$\vec{E}$ & $\vec{H}$ → V & I

Fundamentally, all circuit problems are electromagnetic problems and can be solved as such. All two-conductor transmission lines either support a TEM wave or a wave very closely approximated as TEM. An important property of TEM waves is that $\vec{E}$ is uniquely related to V, and H is uniquely related to I. This reduces analysis of transmission lines to just V and I, making analysis much simpler because these are scalar quantities!



$$V = -\int_L \vec{E} \bullet d\vec{\ell}$$



$$I = \oint_L \vec{H} \bullet d\vec{\ell}$$

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Transmission Line Equations

The transmission line equations do for transmission lines the same thing as Maxwell's curl equations do for electromagnetic waves.

Maxwell's Equations

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$$

$$\nabla \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t}$$

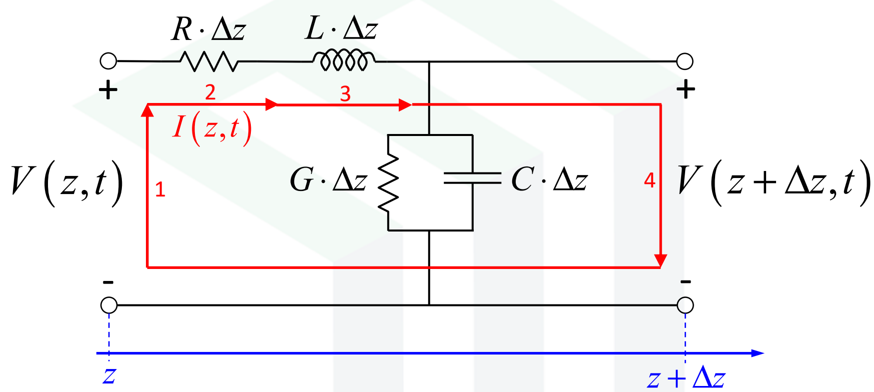
Transmission Line Equations

$$-\frac{\partial V}{\partial z} = RI + L\frac{\partial I}{\partial t}$$

$$-\frac{\partial I}{\partial z} = GV + C\frac{\partial V}{\partial t}$$

Like Maxwell's equations, the transmission line equations are rarely directly useful. Instead, all the useful equations will be derived from them.

Derivation of First TL Equation (1 of 2)



Apply Kirchoff's voltage law (KVL) to the outer loop of the equivalent circuit:

$$\underbrace{-V(z,t)}_{1} + \underbrace{I(z,t)R\Delta z}_{2} + \underbrace{L\Delta z\frac{\partial I(z,t)}{\partial t}}_{3} + \underbrace{V(z+\Delta z,t)}_{4} = 0$$

Derivation of First TL Equation (2 of 2)

Rearrange the last equation by bringing all the voltage terms to the left-hand side of the equation, bringing all the current terms to the right-hand side of the equation, and then dividing both sides by Δz.

$$-V(z,t)+I(z,t)R\Delta z+L\Delta z\frac{\partial I(z,t)}{\partial t}+V(z+\Delta z,t)=0$$

$$\downarrow$$

$$-\frac{V(z+\Delta z,t)-V(z,t)}{\Delta z}=RI(z,t)+L\frac{\partial I(z,t)}{\partial t}$$

In the limit as $\Delta z \rightarrow 0$, the expression on the left-hand side becomes a derivative with respect to z.

$$\boxed{-\frac{\partial V(z,t)}{\partial z}=RI(z,t)+L\frac{\partial I(z,t)}{\partial t}}$$

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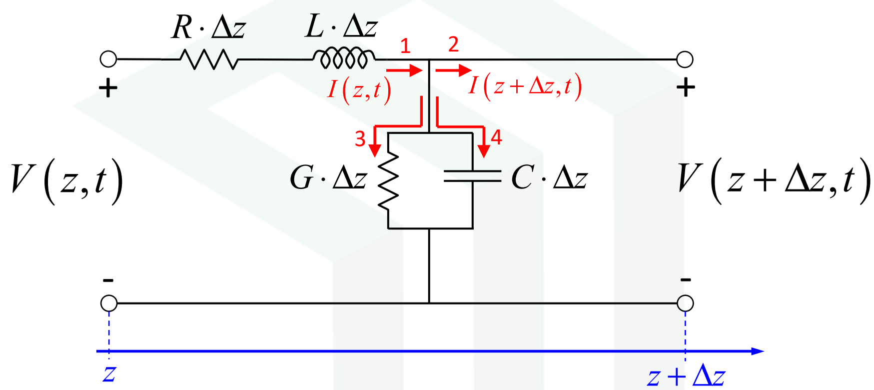
Derivation of Second TL Equation (1 of 2)



Apply Kirchoff's current law (KCL) to the main node the equivalent circuit:

$$\underbrace{I(z,t)}_{1}-\underbrace{I(z+\Delta z,t)}_{2}-\underbrace{G\Delta zV(z+\Delta z,t)}_{3}-\underbrace{C\Delta z\frac{\partial V(z+\Delta z,t)}{\partial t}}_{4}=0$$

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Derivation of Second TL Equation (2 of 2)

Rearrange the last equation by bringing all the current terms to the left-hand side of the equation, bringing all the voltage terms to the right-hand side of the equation, and then dividing both sides by Δz.

$$I(z,t) - I(z+\Delta z,t) - G\Delta z V(z+\Delta z,t) - C\Delta z \frac{\partial V(z+\Delta z,t)}{\partial t} = 0$$

$$\downarrow$$

$$-\frac{I(z+\Delta z,t) - I(z,t)}{\Delta z} = GV(z+\Delta z,t) + C\frac{\partial V(z+\Delta z,t)}{\partial t}$$

In the limit as $\Delta z \rightarrow 0$, the expression on the left-hand side becomes a derivative with respect to z.

$$\boxed{-\frac{\partial I(z,t)}{\partial z} = GV(z,t) + C\frac{\partial V(z,t)}{\partial t}}$$

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Transmission Line Wave Equations

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Starting Point – Telegrapher Equations

Start with the transmission line equations derived in the previous section.

$$-\frac{\partial V(z,t)}{\partial z} = RI(z,t) + L\frac{\partial I(z,t)}{\partial t} \qquad -\frac{\partial I(z,t)}{\partial z} = GV(z,t) + C\frac{\partial V(z,t)}{\partial t} \qquad \text{time-domain}$$

For time-harmonic (i.e. frequency-domain) analysis, Fourier transform the equations above.

$$\boxed{-\frac{dV(z)}{dz} = (R + j\omega L)I(z)} \qquad \boxed{-\frac{dI(z)}{dz} = (G + j\omega C)V(z)} \qquad \text{frequency-domain}$$

Note: The derivative d/dz became an ordinary derivative because z is the only independent variable left.

These last equations are commonly referred to as the *telegrapher equations*.

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Wave Equation in Terms of $V(z)$

$$-\frac{dV(z)}{dz} = (R + j\omega L)I(z) \qquad \text{Eq. (1)}$$

$$-\frac{dI(z)}{dz} = (G + j\omega C)V(z) \qquad \text{Eq. (2)}$$

To derive a wave equation in terms of $V(z)$, first differentiate Eq. (1) with respect to z.

$$-\frac{d^2V(z)}{dz^2} = (R + j\omega L)\frac{dI(z)}{dz} \qquad \text{Eq. (3)}$$

Second, substitute Eq. (2) into the right-hand side of Eq. (3) to eliminate $I(z)$ from the equation.

$$-\frac{d^2V(z)}{dz^2} = -(R + j\omega L)(G + j\omega C)V(z)$$

Last, rearrange the terms to arrive at the final form of the wave equation.

$$\boxed{\frac{d^2V(z)}{dz^2} - (R + j\omega L)(G + j\omega C)V(z) = 0}$$

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Wave Equation in Terms of $I(z)$

$$-\frac{dV(z)}{dz} = (R + j\omega L)I(z) \qquad \text{Eq. (1)}$$

$$-\frac{dI(z)}{dz} = (G + j\omega C)V(z) \qquad \text{Eq. (2)}$$

To derive a wave equation in terms of $I(z)$, first differentiate Eq. (2) with respect to z.

$$-\frac{d^2 I(z)}{dz^2} = (G + j\omega C)\frac{dV(z)}{dz} \qquad \text{Eq. (3)}$$

Second, substitute Eq. (1) into the right-hand side of Eq. (3) to eliminate $V(z)$.

$$-\frac{d^2 I(z)}{dz^2} = -(G + j\omega C)(R + j\omega L)I(z)$$

Last, rearrange the terms to arrive at the final form of the wave equation.

$$\boxed{\frac{d^2 I(z)}{dz^2} - (G + j\omega C)(R + j\omega L)I(z) = 0}$$

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Propagation Constant, γ

In the wave equations, there is the common term $(G + j\omega C)(R + j\omega L)$.

Define the propagation constant γ to be

$$\gamma = \alpha + j\beta = \sqrt{(G + j\omega C)(R + j\omega L)}$$

Given this definition, the transmission line equations are written as

$$\frac{d^2 V(z)}{dz^2} - \gamma^2 V(z) = 0$$

$$\frac{d^2 I(z)}{dz^2} - \gamma^2 I(z) = 0$$

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Solution to the Wave Equations

If the wave equations are handed off to a mathematician, they will return with the following solutions.

$$\frac{d^2V(z)}{dz^2} - \gamma^2 V(z) = 0 \quad \rightarrow \quad V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$$

$$\frac{d^2I(z)}{dz^2} - \gamma^2 I(z) = 0 \quad \rightarrow \quad I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}$$

Forward wave

Backward wave

Both $V(z)$ and $I(z)$ have the same differential equation so it makes sense their solutions have the same mathematical form.

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