Electromagnetics:

Electromagnetic Field Theory

Force Between Two Current Elements

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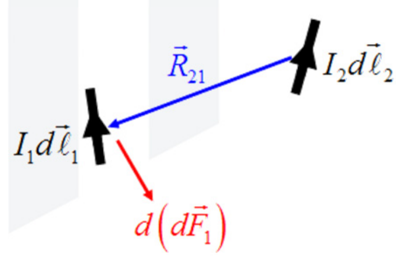
Problem Setup

Consider two current elements, $I_1 d\vec{\ell}_1$ and $I_2 d\vec{\ell}_2$.

Each produces a magnetic field that puts a force on the other.

Let's calculate the force $d(d\vec{F}_1)$ on $I_1 d\vec{\ell}_1$ due to $I_2 d\vec{\ell}_2$.



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Derivation (1 of 2)

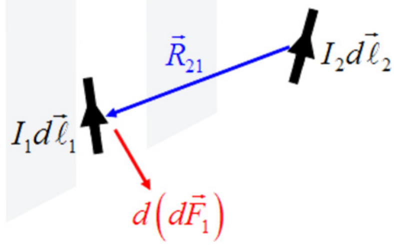
The magnetic field produced by $I_2 d\vec{\ell}_2$ is written using the Biot-Savart law.

$$d\vec{H}_2 = \frac{I_2 d\vec{\ell}_2 \times \hat{a}_{21}}{4\pi R_{21}^2}$$

Write this in terms of $\vec{B}_2$ using the constitutive relation.

$$d\left(\frac{\vec{B}_2}{\mu}\right) = \frac{I_2 d\vec{\ell}_2 \times \hat{a}_{21}}{4\pi R_{21}^2}$$

$$d\vec{B}_2 = \frac{\mu I_2 d\vec{\ell}_2 \times \hat{a}_{21}}{4\pi R_{21}^2}$$



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Derivation (2 of 2)

The force on $I_1 d\vec{\ell}_1$ is

$$d\vec{F}_1 = \left(I_1 d\vec{\ell}_1\right) \times \vec{B}_2$$

We just derived $d\vec{B}_2$.

$$d\vec{B}_2 = \frac{\mu I_2 d\vec{\ell}_2 \times \hat{a}_{21}}{4\pi R_{21}^2}$$

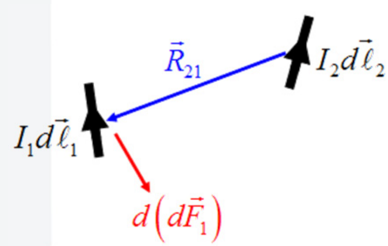
Differentiate this to get

$$d\left(d\vec{F}_1\right) = \left(I_1 d\vec{\ell}_1\right) \times d\vec{B}_2$$

Combine these expressions to get

$$d\left(d\vec{F}_1\right) = \left(I_1 d\vec{\ell}_1\right) \times \frac{\mu I_2 d\vec{\ell}_2 \times \hat{a}_{21}}{4\pi R_{21}^2}$$

$$\boxed{d\left(d\vec{F}_1\right) = \frac{\mu}{4\pi} \frac{\left(I_1 d\vec{\ell}_1\right) \times \left[\left(I_2 d\vec{\ell}_2\right) \times \hat{a}_{21}\right]}{R_{21}^2} = \frac{\mu I_1 I_2}{4\pi} \frac{d\vec{\ell}_1 \times \left(d\vec{\ell}_2 \times \hat{a}_{21}\right)}{R_{21}^2}}$$



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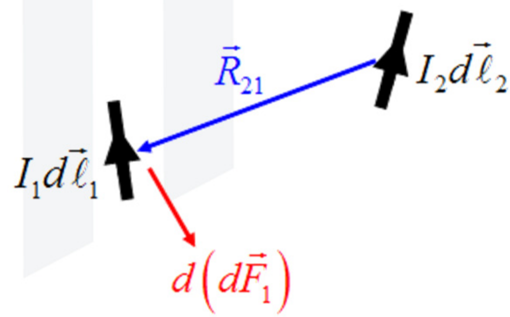
Interpretation

$$d\left(d\vec{F}_1\right) = \frac{\mu}{4\pi}\frac{\left(I_1 d\vec{\ell}_1\right)\times\left[\left(I_2 d\vec{\ell}_2\right)\times\hat{a}_{21}\right]}{R_{21}^2} = \frac{\mu I_1 I_2}{4\pi}\frac{d\vec{\ell}_1\times\left(d\vec{\ell}_2\times\hat{a}_{21}\right)}{R_{21}^2}$$

This equation is analogous to Coulomb's law in electrostatics.

The double differential means we have to integrate over the length of both wires to get the total force.



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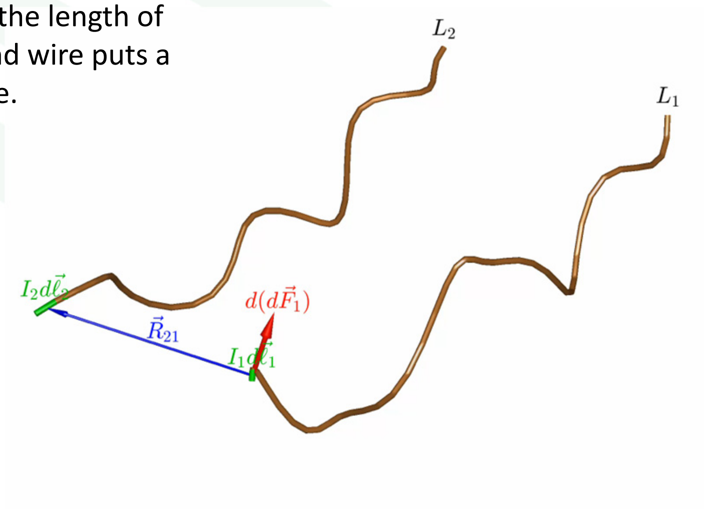
Total Force Between To Line Currents

To calculate the total force between two line currents, we must integrate over the length of each wire. Each part of the second wire puts a force on each part of the first wire.

$$\vec{F}_1 = \int_{L_1}\int_{L_2} d\left(d\vec{F}_1\right)$$

$$\vec{F}_1 = \frac{\mu I_1 I_2}{4\pi}\int_{L_1}\int_{L_2}\frac{d\vec{\ell}_1\times\left(d\vec{\ell}_2\times\hat{a}_{21}\right)}{R_{21}^2}$$



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Example

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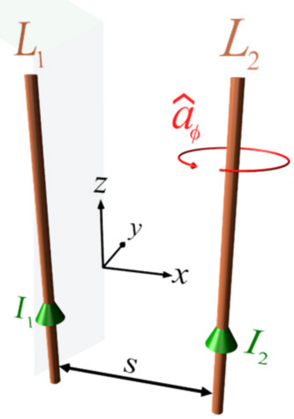
Example #2 – Force Between Two Wires

Two parallel wires carrying currents I_1 and I_2 are a distance s apart.

What is the force per meter between these two wires?



Solution

We will solve this by calculating the magnetic field induced by the second wire and then use this result to calculate the force on the first wire.

Using the infinite wire approximation, the magnetic field around the first wire was found to be

$$\vec{H}_2 = \frac{I_2}{2\pi\rho}\hat{a}_\phi$$

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Example #2 – Force Between Two Wires

Use the constitutive relation to find $\vec{B}$ around the second wire.

$$\vec{H}_2 = \frac{I_2}{2\pi\rho}\hat{a}_\phi \quad \rightarrow \quad \frac{\vec{B}_2}{\mu} = \frac{I_2}{2\pi\rho}\hat{a}_\phi$$

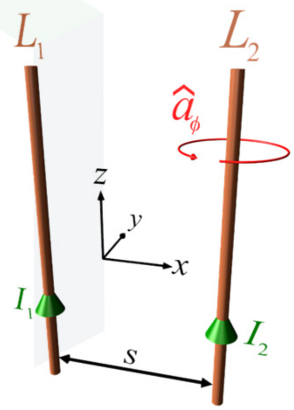
$$\rightarrow \quad \vec{B}_2 = \frac{\mu I_2}{2\pi\rho}\hat{a}_\phi$$

The force this field puts on the first wire is

$$\vec{F}_1 = \int_{L_1}\left(I_1 d\vec{\ell}_1\right)\times\vec{B}_2 = \int_0^L \left(I_1 dz\hat{a}_z\right)\times\left(\frac{\mu I_2}{2\pi\rho}\hat{a}_\phi\right)$$

The coordinate systems have been mixed here.

At the first wire $\hat{a}_\phi = -\hat{a}_y$ and $\rho = s$.



Example #2 – Force Between Two Wires

The equation for force becomes

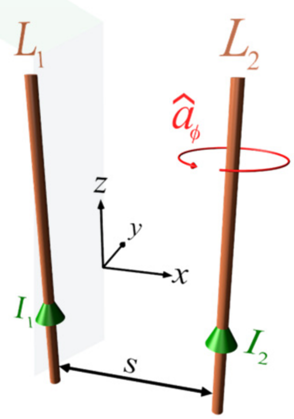
$$\vec{F}_1 = \int_0^L \left(I_1 dz\hat{a}_z\right)\times\left(-\frac{\mu I_2}{2\pi s}\hat{a}_y\right)$$

$$= \frac{\mu I_1 I_2}{2\pi s}\int_0^L \left(\hat{a}_z\right)\times\left(-\hat{a}_y\right)dz$$

$$= \frac{\mu I_1 I_2}{2\pi s}\int_0^L \left(\hat{a}_x\right)dz$$

$$= \frac{\mu I_1 I_2}{2\pi s}\hat{a}_x\int_0^L dz$$

$$= \frac{\mu I_1 I_2}{2\pi s}\hat{a}_x L$$



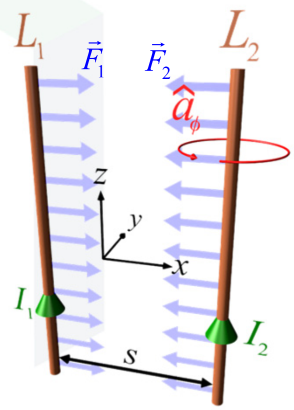
Example #2 – Force Between Two Wires

The force per unit length is

$$\vec{F}_1 = \frac{\mu I_1 I_2}{2\pi s}\hat{a}_x L$$

$$\boxed{\frac{\vec{F}_1}{L} = \frac{\mu I_1 I_2}{2\pi s}\hat{a}_x}$$

Observe that the wires are being attracted toward each other. This is called the *pinch effect*.



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Example #3 – The Numbers

If the wires both carry 1 A, are 1 m long, and are 1 m apart, what is the total force between them?

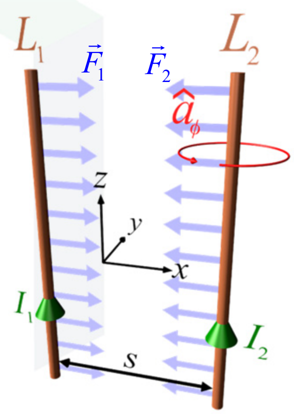
Approximate Solution

We plug these numbers into the equation we just derived.

$$\vec{F}_1 = \frac{\mu_0 \mu_r I_1 I_2 L}{2\pi s}\hat{a}_x$$

$$= \frac{\left(1.2566\times10^{-6}\ \tfrac{\text{H}}{\text{m}}\right)(1.0)(1\ \text{A})(1\ \text{A})(1\ \text{m})}{2\pi(1\ \text{m})}\hat{a}_x$$

$$= \boxed{2.0\times10^{-7}\hat{a}_x\ \text{N} = 200\hat{a}_x\ \text{nN}}$$



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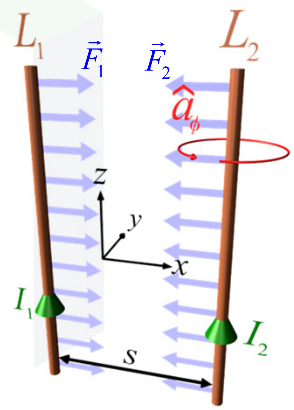
Example #4 – Exact Numbers

If the wires both carry 1 A, are 1 m long, and are 1 m apart, what is the total force between them?



Exact Solution

The rigorous equation to calculate the force between these two wires is

$$\vec{F}_1 = \frac{\mu I_1 I_2}{4\pi} \int_{L_1} \int_{L_2} \frac{d\vec{\ell}_1 \times \left(d\vec{\ell}_2 \times \hat{a}_{21} \right)}{R_{21}^2}$$

$$= \frac{\mu I_1 I_2}{4\pi} \int_{L_1} \int_{L_2} d\vec{\ell}_1 \times \left(d\vec{\ell}_2 \times \frac{\vec{R}_{21}}{\left| \vec{R}_{21} \right|^3} \right)$$

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Example #4 – Exact Numbers

Determine expressions for each term in the force equation.

$$\vec{F}_1 = \frac{\mu I_1 I_2}{4\pi} \int_{L_1} \int_{L_2} d\vec{\ell}_1 \times \left(d\vec{\ell}_2 \times \frac{\vec{R}_{21}}{\left| \vec{R}_{21} \right|^3} \right)$$

$$\mu = \mu_0$$

$$I_1 = 1 \text{ A}$$

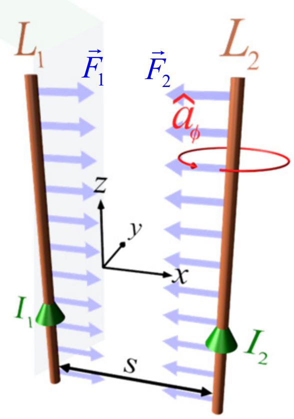
$$I_2 = 1 \text{ A}$$

$$d\vec{\ell}_1 = dz_1 \hat{a}_z$$

$$d\vec{\ell}_2 = dz_2 \hat{a}_z$$

$$\vec{R}_{21} = -s\hat{a}_x + (z_2 - z_1)\hat{a}_z$$

$$\frac{\vec{R}_{21}}{\left| \vec{R}_{21} \right|^3} = \frac{-s\hat{a}_x + (z_2 - z_1)\hat{a}_z}{\left[s^2 + (z_2 - z_1)^2 \right]^{3/2}}$$



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Example #4 – Exact Numbers

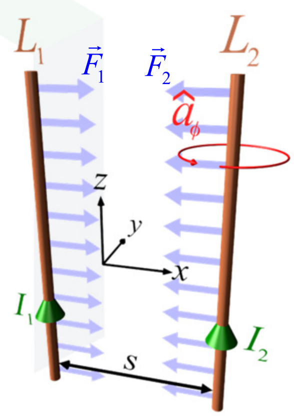
The force equation becomes

$$\vec{F}_1 = \frac{\mu I_1 I_2}{4\pi} \int_{L_1}\int_{L_2} d\vec{\ell}_1 \times \left(d\vec{\ell}_2 \times \frac{\vec{R}_{21}}{\left|\vec{R}_{21}\right|^3} \right)$$

$$= \frac{\mu I_1 I_2}{4\pi} \int_0^L\int_0^L dz_1 \hat{a}_z \times \left(dz_2 \hat{a}_z \times \frac{-s\hat{a}_x + (z_2 - z_1)\hat{a}_z}{\left[s^2 + (z_2 - z_1)^2\right]^{3/2}} \right)$$

$$= \frac{\mu I_1 I_2}{4\pi} \int_0^L\int_0^L dz_1 \hat{a}_z \times \left(\frac{-s dz_2}{\left[s^2 + (z_2 - z_1)^2\right]^{3/2}} \hat{a}_y \right)$$

$$= \frac{\mu s I_1 I_2}{4\pi} \hat{a}_x \int_0^L\int_0^L \frac{dz_1 dz_2}{\left[s^2 + (z_2 - z_1)^2\right]^{3/2}}$$



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Example #4 – Exact Numbers

If we $L \rightarrow \infty$, we get the approximate result from the last example.

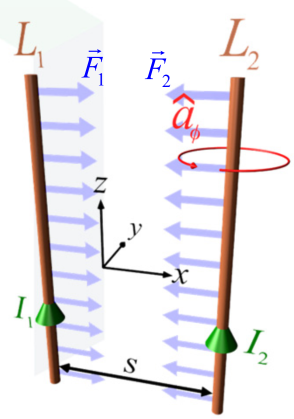
$$\vec{F}_1 = \frac{\mu_0 \mu_r I_1 I_2 L}{2\pi s} \hat{a}_x \rightarrow (200 \text{ nN}) \hat{a}_x$$

Performing the integration gives us the exact result of

$$\boxed{\vec{F}_1 = (82.8 \text{ nN}) \hat{a}_x}$$

If the first wire were 1 m long and the second was 10 m long, the force would be

$$\vec{F}_1 = (196 \text{ nN}) \hat{a}_x$$



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