Computational Science:
Computational Methods in Engineering

Intersection of a Ray Crossing a Triangle

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Outline

- Barycentric Coordinates
- Intersection of Ray with Triangle

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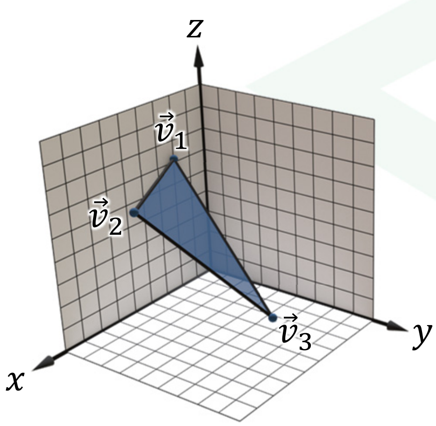
Barycentric Coordinates

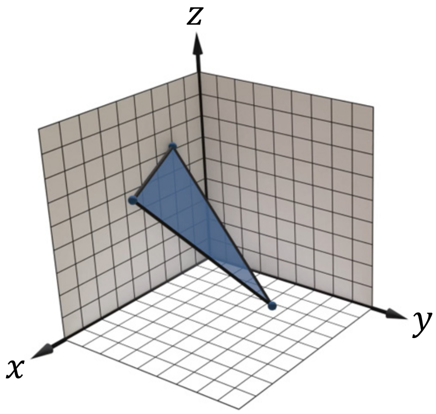
Basics of a Triangle



A triangle may exist with its three vertices distributed at any positions in three-dimensional space.

$$\vec{v}_1 = (v_{1,x}, v_{1,y}, v_{1,z})$$
$$\vec{v}_2 = (v_{2,x}, v_{2,y}, v_{2,z})$$
$$\vec{v}_3 = (v_{3,x}, v_{3,y}, v_{3,z})$$

A More Convenient View of the Triangle



While the actual position and orientation of the triangle is not changing, it is more convenient to look at the triangle straight on.

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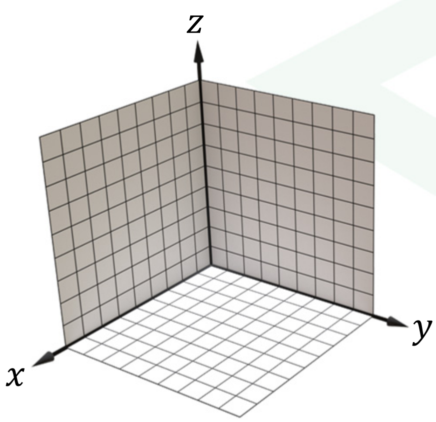
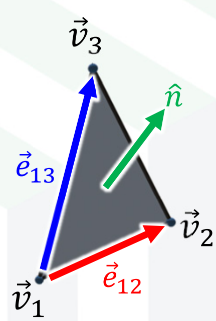
Edge Vectors and Normal Vector





Define the edge vectors.

$$\vec{e}_{12} = \vec{v}_2 - \vec{v}_1$$
$$\vec{e}_{13} = \vec{v}_3 - \vec{v}_1$$

From the edge vectors, the surface normal is calculated.

$$\hat{n} = \vec{e}_{12} \times \vec{e}_{13}$$

Observe the order of the vertices also conveys the direction of the normal vector following the right-hand rule.

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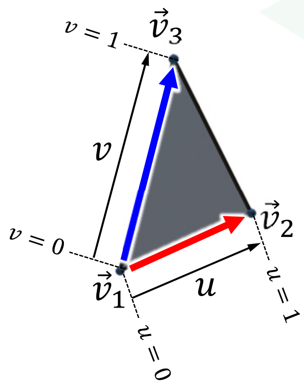
Definition of Barycentric Coordinates (u, v)



All points within the plane of the 3D triangle can be identified from just two scalar quantities called the *Barycentric coordinates* (u, v).

$$\vec{p} = \vec{v}_1 + u\vec{e}_{12} + v\vec{e}_{13}$$

Points within the triangle are identified as

$$0 \le u \le 1 \qquad 0 \le v \le 1 \qquad u + v \le 1$$

All other values of (u, v) lie within the plane of the triangle, but outside of the triangle itself.

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Intersection of Ray with Triangle

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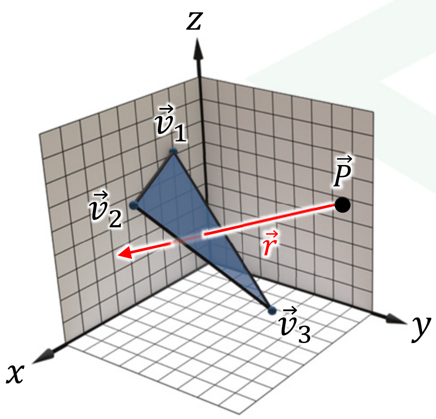
Geometry of the Problem



Start with the 3D triangle.

Now define a point $\vec{P}$ and ray direction $\vec{r}$.

Any point along the ray can be written as

$$\vec{p}(\alpha) = \vec{P} + \alpha\vec{r}$$

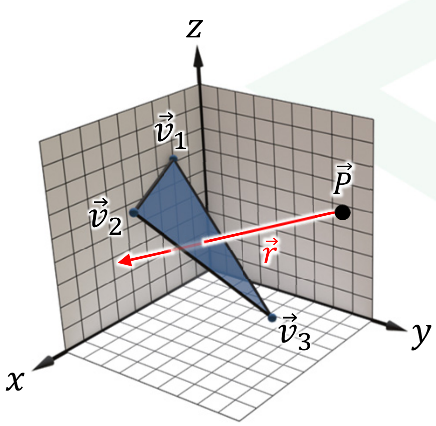
Any point in the plane of the triangle can be written as

$$\vec{p}(u,v) = \vec{v}_1 + u\vec{e}_{12} + v\vec{e}_{13}$$

Setting these equal $\vec{p}(\alpha) = \vec{p}(u,v)$ gives

$$\vec{P} + \alpha\vec{r} = \vec{v}_1 + u\vec{e}_{12} + v\vec{e}_{13}$$

Formulation of Matrix Equation



Starting with

$$\vec{P} + \alpha\vec{r} = \vec{v}_1 + u\vec{e}_{12} + v\vec{e}_{13}$$

The equation is rearranged to put all the unknowns on the left and all the knowns on the right.

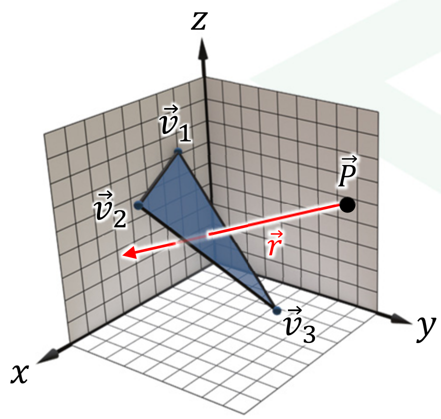
$$-\alpha\vec{r} + u\vec{e}_{12} + v\vec{e}_{13} = \vec{P} - \vec{v}_1$$

In matrix form this is

$$\begin{bmatrix} -\vec{r} & \vec{e}_{12} & \vec{e}_{13} \end{bmatrix} \begin{bmatrix} \alpha \\ u \\ v \end{bmatrix} = \begin{bmatrix} \vec{P} - \vec{v}_1 \end{bmatrix}$$

$$\begin{bmatrix} -r_x & e_{12,x} & e_{13,x} \\ -r_y & e_{12,y} & e_{13,y} \\ -r_z & e_{12,z} & e_{13,y} \end{bmatrix} \begin{bmatrix} \alpha \\ u \\ v \end{bmatrix} = \begin{bmatrix} P_x - v_{1,x} \\ P_y - v_{1,y} \\ P_z - v_{1,z} \end{bmatrix}$$

Numerical Solution



The solution is solved according to

$$\begin{bmatrix} \alpha \\ u \\ v \end{bmatrix} = \begin{bmatrix} -r_x & e_{12,x} & e_{13,x} \\ -r_y & e_{12,y} & e_{13,y} \\ -r_z & e_{12,z} & e_{13,y} \end{bmatrix}^{-1} \begin{bmatrix} P_x - v_{1,x} \\ P_y - v_{1,y} \\ P_z - v_{1,z} \end{bmatrix}$$

Given the solution, the point where the ray crosses the plane of triangle can be calculated in either of two ways.

$$\vec{p}(\alpha) = \vec{P} + \alpha\vec{r}$$

$$\vec{p}(u,v) = \vec{v}_1 + u\vec{e}_{12} + v\vec{e}_{13}$$

The point lies within the triangle if all of the following conditions are satisfied.

$$0 \le u \le 1 \qquad 0 \le v \le 1 \qquad u + v \le 1$$

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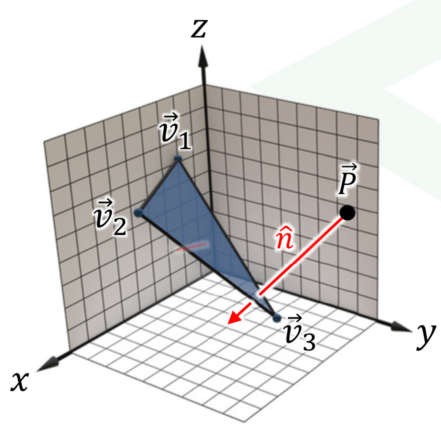
Distance from Point to Triangle (1 of 2)



Step 1 – Let the ray direction be in the direction of the normal vector. This ensures shortest distance from the point to the plane of the triangle.

$$\vec{r} = \hat{n}$$

Step 2 – Calculate the intersection on the plane of the triangle.

$$\begin{bmatrix} \alpha \\ u \\ v \end{bmatrix} = \begin{bmatrix} -n_x & e_{12,x} & e_{13,x} \\ -n_y & e_{12,y} & e_{13,y} \\ -n_z & e_{12,z} & e_{13,y} \end{bmatrix}^{-1} \begin{bmatrix} P_x - v_{1,x} \\ P_y - v_{1,y} \\ P_z - v_{1,z} \end{bmatrix}$$

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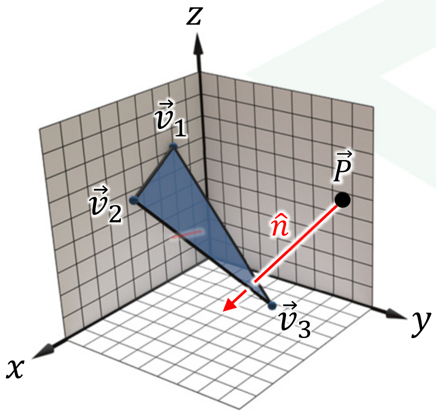
Distance from Point to Triangle (3 of 2)



Step 3 – If the point lies outside triangle, determine what point in triangle is the closest.